MATH 1113
Final Exam Solution

1. Let $f(x)=\dfrac{\sqrt{1-x}}{x+9}$.

a. State the domain of f.

$\{x|x \leq 1 \text{ and } x \neq -9\}$ or $(-\infty,-9) \cup (-9,1]$

b. Find *f*(-3).

$f(-3) = \frac{\sqrt{1-(-3)}}{(-3)+9} = \frac{\sqrt{4}}{6} = \frac{2}{6} = \frac{1}{3}$

2. Given $f(x)=x^2$, find the average rate of change of f from $x_1 = 3$ to $x_2 = 3+h$. Express your answer in simplest form.

$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= \frac{(3+h)^2-3^2}{h}\\ &= \frac{9+6h+h^2-9}{h}\\ &= \frac{h(6+h)}{h}\\ &= 6+h, \qquad h \neq 0\end{aligned}$$

3. The polynomial function $P(x) = x^3 - 6x^2 + 11x - 6$ has only real zeros.

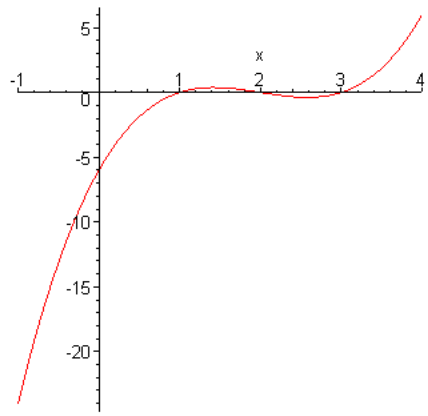
a. Use the factor theorem to determine if 2 a zero of this function.

$P(2) = 2^3 - 6 \cdot 2^2 + 11 \cdot 2 - 6 = 8 - 24 + 22 - 6 = 0$, *so yes it is.*

b. Find the end-behavior of P(x), and sketch its graph.

End behavior:
As $x \to -\infty$, $P(x) \to -\infty$.
As $x \to +\infty$, $P(x) \to +\infty$.

Graph:



c. Express *P(x)* as a product of linear factors.

$$P(x) = (x - 1)(x - 2)(x - 3)$$

4. The Intermediate Value Theorem for Polynomial Functions: If *f* is a polynomial function and $[a, b]$ is a closed interval from the domain of *f*, then *f* takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

a. Use the intermediate value theorem to show that $f(x) = x^3 - 2$ has a zero between $x = 1$ and $x = 2$.

Since $f(1) = -1$ *and* $f(2) = 6$ *and 0 is between –1 and 6, the Intermediate Value Theorem guarantees there is a number c* $\in [0, 1]$ *such that* $f(c) = 0$.

b. Use your calculator to estimate the zero to at least two decimal places of accuracy.

$f(\sqrt[3]{2}) = 0$ *and* $\sqrt[3]{2} \approx 1.26$

5. Given the rational function $r(x) = \dfrac{x^2 + 3x - 4}{x^2 - 9}$

a. State the domain of *r*(x).

All real numbers except –3 and 3 or $\{x | x \neq -3,\ x \neq 3\}$ *or* $(-\infty, -3) \cup (-3, 3) \cup (3, +\infty)$

b. Find the vertical asymptotes of the graph of *r*(x).

$x = -3$ *and* $x = 3$

c. Find the x -intercepts of the graph of *r*(x).

$(-4, 0)$ *and* $(1, 0)$

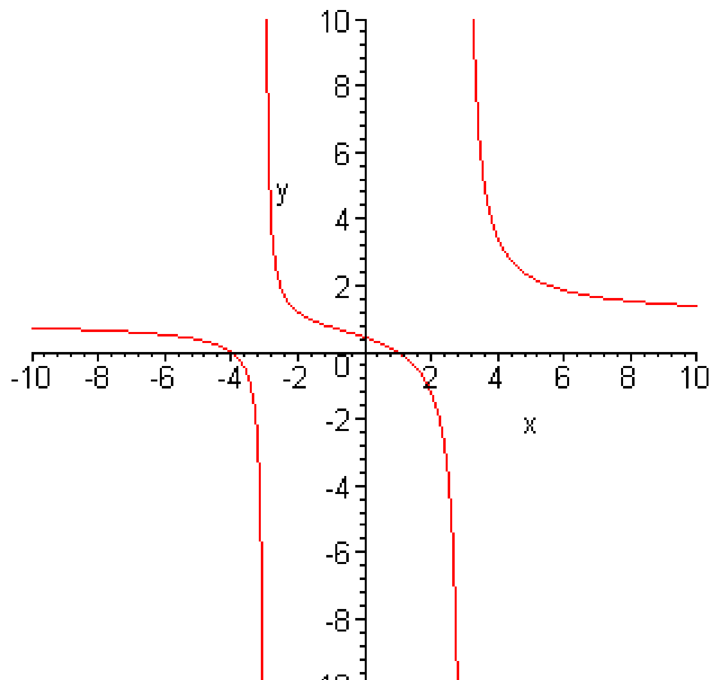
d. Find the y -intercepts of the graph of *r*(x).

$\left(0, \frac{4}{9}\right) \approx (0, 0.444)$

e. Find the horizontal asymptote of the graph of *r*(x).

$y = 1$

f. Sketch the graph of r(x).



6. Solve the equation. Leave your answer in exact form.

$$3^{x+1} = 2^{2x}$$

$$\ln 3^{x+1} = \ln 2^{2x}$$

$$(x+1)\cdot\ln 3 = 2x\cdot\ln 2$$

$$x\cdot\ln 3 + \ln 3 = 2x\cdot\ln 2$$

$$\ln 3 = 2x\cdot\ln 2 - x\cdot\ln 3$$

$$\ln 3 = x(2\ln 2 - \ln 3)$$

$$x = \frac{\ln 3}{2\ln 2 - \ln 3}$$

Note: This solution has many equivalent forms.

7. Solve the equation. Leave your answer in exact form.

$$\ln(x) - \ln(x-1) = 1$$

$$\ln\frac{x}{x-1} = 1$$

$$\frac{x}{x-1} = e^1$$

$$x = e(x-1)$$

$$x = ex - e$$

$$e = ex - x$$

$$e = x(e-1)$$

$$x = \frac{e}{e-1}$$

8. The amount of Carbon 14 (in grams) in a piece of charcoal from a tree killed during an ancient volcanic eruption has been shown by experiment to be

$$Q(t) = 3e^{\frac{-\ln 2}{5730}t}$$

a. Estimate $Q(t)$ if t = 9000 years. (Round your answer to the nearest hundredth of a gram)

$$Q(9000) = 3e^{\left(\frac{-\ln 2}{5730}9000\right)} \approx 1.01\ g$$

b. Estimate the time t for which $Q(t)$ = 2 grams.

$2 = 3e^{\left(\frac{-\ln 2}{5730}t\right)}$, *so solving for t gives* $t = \frac{-5730\ln\left(\frac{2}{3}\right)}{\ln 2} \approx 3300\ years$

9. Let $f(x) = 1 + \frac{3}{x}$ and $g(x) = x^2 + 1$.

a. Find the inverse function of f.

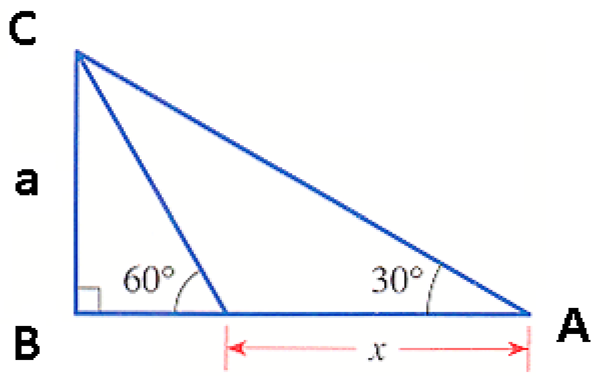
$$y = 1 + \frac{3}{x}$$
$$x = 1 + \frac{3}{y}$$
$$x - 1 = \frac{3}{y}$$
$$\frac{1}{x-1} = \frac{y}{3}$$
$$\frac{3}{x-1} = y$$

So $f^{-1}(x) = \frac{3}{x-1}$

b. Find $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 1 + \frac{3}{x^2 + 1}$$

10. If **a**= 20 in the following figure, find the exact value of *x*.



$$x = \frac{40}{\sqrt{3}}$$

Note: There are many approaches that will solve this problem. All of them use trigonometry and geometry to figure out lengths of sides and measures of angles.

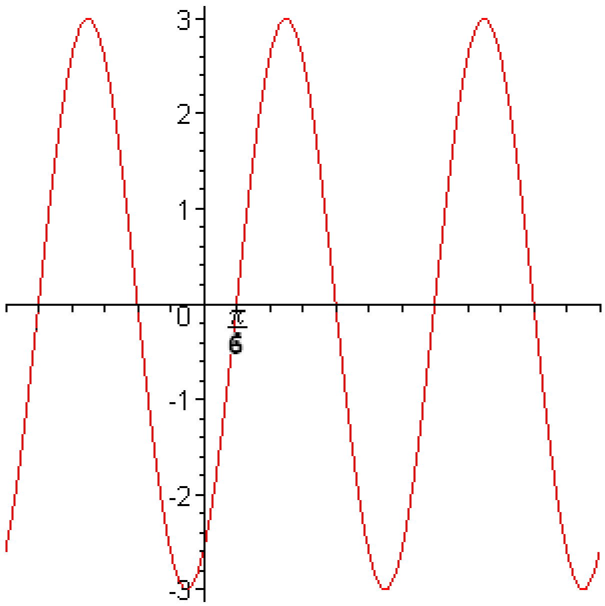
11. From a point 5 meters above level ground, an observer measures the angle of depression to an object on the ground is 69°. Approximate the distance from the object to the point on the ground directly beneath the observer. Round your answer to the nearest tenth of a meter.

$\tan 69° = \frac{5}{x}$, *where x is the distance in question. Thus,* $x = \frac{5}{\tan 69°} \approx 1.9\ m$

12. Find the sign of the expression cot(*t*)tan(*t*) if the terminal point determined by *t* is in the quadrant II.

In quadrant II, $cot(t)$*is negative and* $tan(t)$ *is negative. So* $cot(t) \cdot tan(t)$ *is (negative)(negative) =* ***positive***.
Or note that $cot(t) \cdot tan(t) = 1$, *which is* ***positive****.*

13. In the following graph of a sine curve, the vertical scale is as shown and the horizontal scale is $\frac{\pi}{6}$.



a.) Determine the amplitude of the curve.

The amplitude is 3.

b.) Determine the period of the curve.

The period is π.

c.) Determine the phase shift of the curve.

The phase shift is $\frac{\pi}{6}$.

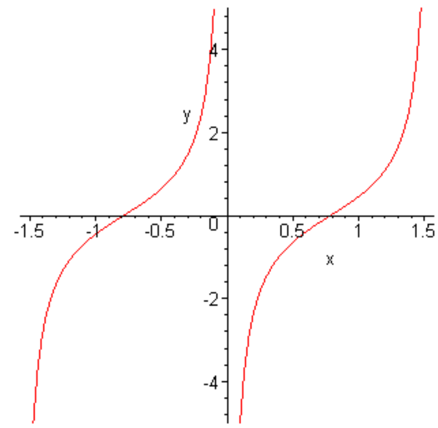
d.) Determine the function in the form $f(x) = a\sin(k(x-b))$ or $f(x) = a\sin(bx+c)$.

$$f(x) = 3\sin 2\left(x - \frac{\pi}{6}\right) = 3\sin\left(2x - \frac{\pi}{3}\right)$$

14. Sketch 2 periods of the graph of $y = \tan\left(2x + \frac{\pi}{2}\right)$. Label the asymptotes and the x-intercepts.

Asymptotes on this graph: $x = \frac{-\pi}{2}, x = 0, x = \frac{\pi}{2}$

X-intercepts on this graph: $\left(\frac{-\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right)$



15. Verify the following trigonometric identity.

$$\frac{\sec x - \cos x}{\tan x} = \sin x$$

$$\begin{aligned}
\frac{\sec x - \cos x}{\tan x} &= \frac{\frac{1}{\cos x} - \cos x}{\frac{\sin x}{\cos x}} \\
&= \left(\frac{1}{\cos x} - \cos x\right) \cdot \frac{\cos x}{\sin x} \\
&= \frac{1 - \cos^2 x}{\sin x} \\
&= \frac{\sin^2 x}{\sin x} \\
&= \sin x
\end{aligned}$$

16. Find all solutions of the following equation in the interval $[0, 2\pi)$. Leave your answers in exact form.

$$2\cos^2 x + \cos x = 1$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

$$2\cos x - 1 = 0 \ or \ \cos x + 1$$

$$\cos x = \frac{1}{2} \ or \ \cos x = -1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \ or \ x = \pi$$

17. Find the solutions of the following equation that are in the interval $[0, 2\pi)$. Assume θ is measured in radians and round any answers correct to 2 decimal places.

$$5\cos\theta - 3 = 0$$

$\cos\theta = \frac{3}{5}$, *so one solution is* $\theta = \cos^{-1}\frac{3}{5} \approx 0.93$
The second solution is $\theta = 2\pi - \cos^{-1}\frac{3}{5} \approx 5.36$

18. Find the exact value of the following expressions.

a. $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{1}{\sqrt{8}}$

Note: To solve, make a right triangle with an angle who's sine is $\frac{1}{3}$, *then use the triangle to figure out the tangent of that angle.*

b. $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

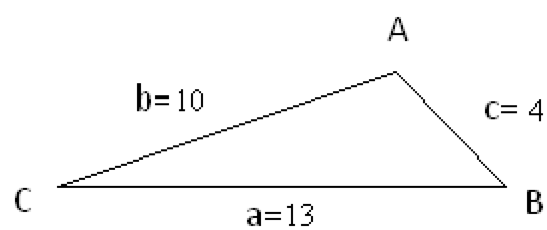
19. Given $\sec(\theta) = -\frac{5}{4}$ *and* $\pi \le \theta \le \frac{3\pi}{2}$, find exact values for the following expressions.

Thanks to the given information, we know $\cos\theta = \frac{-4}{5}$ *and* $\sin\theta = \frac{-3}{5}$.

a. $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{-3}{5}\right)\left(\frac{-4}{5}\right) = \frac{24}{25}$

b. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{-4}{5}}{2}} = \frac{-1}{\sqrt{10}}$

20. Use the law of sines or the law of cosines to find the angles in the following triangle. State the angles in degrees to the nearest hundredth of a degree.



$$13^2 = 10^2 + 4^2 - 2(10)(4)\cos A$$

$$169 = 116 - 80\cos A$$

$$\frac{169 - 116}{-80} = \cos A$$

$$A = \cos^{-1}\frac{-53}{80} \approx 131.49°$$

$$\frac{\sin C}{4} = \frac{\sin 131.49°}{13}$$

$$\sin C = \frac{4 \cdot \sin 131.49°}{13} \approx 0.23048$$

$$C = \sin^{-1}(0.23048) \approx 13.32°$$

$$B \approx 180° - 131.49° - 13.32° = 35.19°$$

Addition/Subtraction Identities

$\sin(u+v) = \sin u \cos v + \cos u \sin v$

$\sin(u-v) = \sin u \cos v - \cos u \sin v$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$\cos(u-v) = \cos u \cos v + \sin u \sin v$

Double Angle Identities

$\sin 2u = 2 \sin u \cos u$

$\cos 2u = \cos^2 u - \sin^2 u$

$\cos 2u = 2\cos^2 u - 1$

$\cos 2u = 1 - 2\sin^2 u$

Half Angle Identities

$\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$

$\cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$

Law of Sines

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where *a, b, c* are lengths of sides and *A, B, C* are the opposite angles.

Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$, where *a, b, c* are lengths of sides and *A* is the angle opposite side *a*.