From Tessellations Big

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IN 1994, THE WORLD CUP SOCCER CHAMPIONships were held in various cities and in a variety of stadiums across the United States. Unlike American football, soccer is played almost exclusively on natural grass. The need for grass presented a problem for Detroit because its stadium, the Silverdome, is an indoor field with artificial turf. Growing grass in domed stadiums has not yet been successful, so the World Cup organizers turned to the soil scientists at Michigan State University. The scientists grew the grass outdoors on large pallets and then moved these pallets into the stadium in time for the games.

What shape would allow such pallets to fit together most efficiently? This article looks at this connection between soccer fields and mathematics to help students understand and enjoy some middle school geometry topics. The activities described in this article can help students see the connections between tessellations and the measures of the interior angles of polygons, between tessellations and regular polyhedra, and between regular polyhedra and semiregular polyhedra.

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Regular Tessellations and Interior Angles

FIGURE 1'S ACTIVITY WILL GIVE STUDENTS AN excellent opportunity to understand the measures of the interior angles of the polygon. At the conclusion of this activity, you can introduce the soccer-field example. The shape that the soil scientists chose to use was a hexagon. Students can discuss why the scientists might have chosen a hexagon over the other regular polygonal shapes. For further information about the grass tessellation of the Silverdome, see the Web site at **web.msu.edu/turf/**.

As explained in **figure 1,** a tessellation is an arrangement of shapes that will cover an entire plane, leaving no gaps and having none of the shapes overlap. In this article we consider only tes-

to Polyhedra:

sellations made with polygonal shapes, and we assume that all polygons are regular.

Students work with card-stock copies of triangles, squares, pentagons, hexagons, heptagons, octagons, nonagons, decagons, and dodecagons. It is helpful if the triangles are all of the same color; the squares, of a second color; and so on. It is important that the lengths of the sides of all the polygons be the same, as shown on the **worksheet** titled "One-Inch Stencils."As students begin the Tessellations activity, they should each have an envelope containing about ten triangles, squares, pentagons, and hexagons and about five of each of the rest of the polygons.

Individuals or groups can do this activity, but it is important that students write their descriptions using such correct terminology as *obtuse* and *acute angles*. Introducing these terms leads to the underlying concept of the size of the interior angles of the polygons.

With help, the students will conclude that the polygons must fit around a single point and that the fit depends on the size of the interior angles. Students also need to know that the sum of the measures of the angles around that single point is 360 degrees, a concept used for the second activity.

The topic of the Polygons and Angles activity is the interior angles of regular polygons (see **fig. 2**).

Tessellations

To *tessellate* means to completely cover a flat surface with one or more types of shapes so that no gaps or overlaps occur. Other words used to describe tessellations are *tilings* and *mosaics*.

Which regular polygons will tessellate by themselves?

Polygon	Does It Tessellate?	If Not, Describe Why Not
Triangle		
Square		
Pentagon		
Hexagon		
Heptagon		
Octagon		
Nonagon		
Decagon		
Dodecagon		

Briefly explain why you think that some polygons tessellate and some do not.

Fig. 1 An activity that explores tessellations

For this activity, students must understand that the sum of the interior angles of any triangle is 180 degrees. The problem-solving approach to finding the interior-angle measures of regular polygons motivates students better than using a specific algorithm.

Ask your students, "What is the measure of an interior angle of a regular pentagon?" As they work on this question, you might prompt them with such questions as these:

- "What is the measure of an interior angle of an equilateral, or regular, triangle?"
- "How would you calculate the angle measure of a regular triangle?"

Polygons and Angles

Since the key to a tessellation is getting the angles of the polygons to fit around a single point, we need to gain a better understanding of how big are the interior angles of the various polygons.

1. What is the measure of an interior angle of a regular pentagon?

2. Write a description of the method used to find the angle measure of the regular pentagon.

3. What is the measure of an interior angle of a regular hexagon? Can the method described for pentagons be used for hexagons?

4. Write a description of a method that can be used to find the angle measure of one angle in any regular polygon.

5. Use the method described above to complete the table below.

Polygon	Number of Sides	Calculations	Measure of Each Angle
Triangle	3		
Square	4		
Pentagon			
Hexagon			
Heptagon			
Octagon			
Nonagon			
Decagon			
Dodecagon			

Fig. 2 Students explore polygons and angles

TABLE 1
Complete List of Combinations of Two or More Regular Polygons Arranged So That Adjacent Angles Sum to 360°

Polygon Combination	Sum of Angles
4 triangles, 1 hexagon	$60° + 60° + 60° + 60° + 120° = 360°$
3 triangles, 2 squares	$60° + 60° + 60° + 90° + 90° = 360°$
2 triangles, 2 hexagons	$60° + 60° + 120° + 120° = 360°$
2 triangles, 1 square, 1 dodecagon	$60° + 60° + 90° + 150° = 360°$
1 triangle, 2 dodecagons	$60° + 150° + 150° = 360°$
1 triangle, 2 squares, 1 hexagon	$60° + 90° + 90° + 120° = 360°$
1 triangle, 1 heptagon, 1 42-gon	$60° + 128\frac{4°}{7} + 171\frac{3°}{7} = 360°$
1 triangle, 1 octagon, 1 24-gon	$60° + 135° + 165° = 360°$
1 triangle, 1 nonagon, 1 18-gon	$60° + 140° + 160° = 360°$
1 triangle, 1 decagon, 1 15-gon	$60° + 144° + 156° = 360°$
1 square, 2 octagons	$90° + 135° + 135° = 360°$
1 square, 1 pentagon, 1 20-gon	$90° + 108° + 162° = 360°$
1 square, 1 hexagon, 1 dodecagon	$90° + 120° + 150° = 360°$
2 pentagons, 1 decagon	$108° + 108° + 144° = 360°$

- "Would it help to know the sum of all the angles in a regular pentagon?"
- "Do you think that you could use the fact that the sum of the interior angles of a triangle is 180 degrees?"
- "Could you use the fact that the sum of the angles around any given point is 360 degrees?"

After the students find the measure of the interior angles of a regular pentagon, have them write a description of the method that they used to calculate this angle measure. One approach that is often invented by students is this: (1) Draw a point in the interior of the polygon. (2) Construct segments from that point to each vertex of he polygon. (3) Multiply the number of triangles created by 180 degrees. (4) Subtract 360 degrees for the extra angles created around the point in the interior. (5) Divide the difference, which is the sum of the interior angles, by the number of angles (see **fig. 3**).

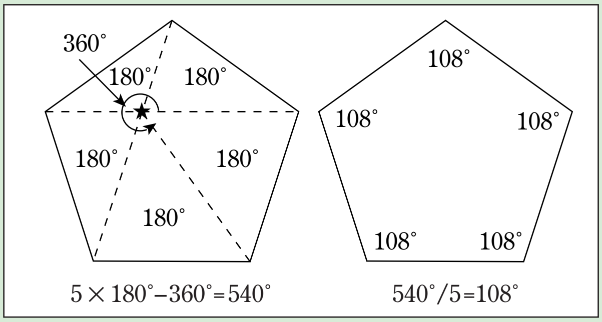


Fig. 3 An uncommon yet intuitive method

With their invented algorithm, the students can fill in the table at the bottom of **figure 2.** With this list of angle measures, the students should look back at the Tessellations activity to see whether they notice a relationship between the regular polygons that will tessellate and their angle measures. When students see that the only polygons that will tessellate by themselves are those whose angle measures evenly divide 360 degrees, I have had them respond with an audible "Aha."

The previous two activities can be completed in about one hour.

Other Tessellations with Regular Polygons

Now that we understand that one of the keys to creating a tesselation is to have the sum of the angles of the polygons around a point be 360 degrees, we can use the angle measures determined in the Polygons and Angles activity in **figure 2** to find combinations of more than one type of regular polygon that fit around a single point.

1. Use the angle measures determined in the previous activity to find different combinations of at least two different types of polygons with angles that add to 360 degrees.

2. Use the polygons that you cut out to see how each of the combinations of polygons can be arranged around a single point, and sketch this arrangement in the space provided at the right in the table.

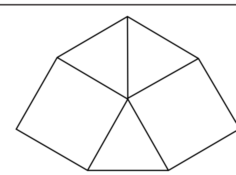
List of Polygons Angle Sum Check	Sketch of How the Polygons Are Arranged
3 triangles and 2 squares	
$60° + 60° + 60° + 90° + 90° = 360°$, or $3 \times 60° + 2 \times 90° = 360°$	

Fig. 4 A next step in understanding tessellations

Angles and Other Tessellations

TESSELLATIONS LIKE THOSE IN **FIGURE 1,** which require only one type of regular polygon, are called *regular* tessellations. A logical follow-up to the study of regular tessellations is the question "Do any combinations exist of more than one type of polygon whose angles sum to 360 degrees?" Using the table created in the previous activity, ask your students to use more than one type of regular polygon to find angle combinations that sum to 360 degrees and to record their findings on Other Tessellations with Regular Polygons (see **fig. 4**). The students can find these combinations by arranging the polygon cutouts around a single point and then checking the angle measures, or vice versa. For example, two dodecagons with an interior angle of 150 degrees and a triangle with an interior angle of 60 degrees will fit around a point because 150° + 150° + 60° = 360°. By carefully checking cases, one can see that **table 1** on page 351 contains a list of all possible combinations (Seymour and Britton 1989, 52). Finding all possible combinations on the list is a nice challenge for some students.

"Is only one arrangement possible for each combination of polygons?" is a question that arises naturally. An example of more than one arrangement of three triangles and two squares is seen in **figure 5.** Encourage students to determine which other polygon combinations in their list have more than one arrangement. The key factor in determining when one arrangement of polygons is the same as another is examining which polygons are adjacent.

In addition to the case of three triangles and two squares, three other combinations of polygons have two different arrangements around a single point. Thus the fourteen polygon combinations with adjacent angles that sum to 360 degrees yield eighteen different arrangements of two or more polygons around a single point (Seymour and Britton 1989, 52).

To construct a tessellation, is it sufficient to have a combination of regular polygons whose adjacent angle measures sum to 360 degrees? The answer depends on what restrictions we put on the tessellation itself. Let us consider *semiregular* tessellations, which are defined as tessellations using more than one type of regular polygon, in which the arrangement of polygons is the same at every point. Referring to **table 1,** the second combination with the arrangement of 3-3-4-3-4 forms a semiregular tessellation (see **fig. 6a**). The last combination of 5-5-10 does not form a tessellation at all, as **figure 6c** shows. The example shown in **figure 6b** is not a semiregular tessellation because the arrangement of polygons is not the same at every vertex. The arrangement at vertex A is 6-4-3-4 and at vertex B is 6-4-4-3. These examples illustrate that even though the combination of angles sums to 360 degrees, we may not be able to construct a semiregular tessellation. In fact, the eighteen possible arrangements of regular polygons around a point yields only eight semiregular tessellations (Seymour and Britton 1989, 52), which are shown in **figure 7.**

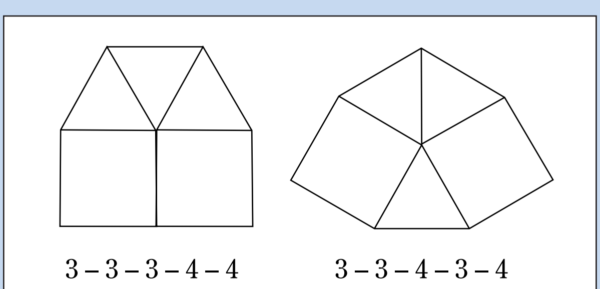


Fig. 5 Two arrangements of the same five polygons

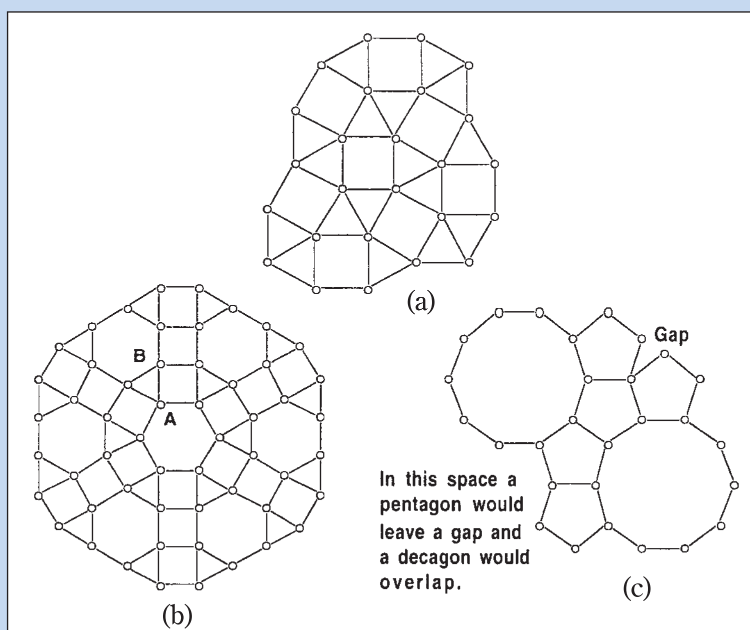


Fig. 6 Examples of a tessellation that is semiregular (a), one that is not (b), and an arrangement that is not a tessellation (c)

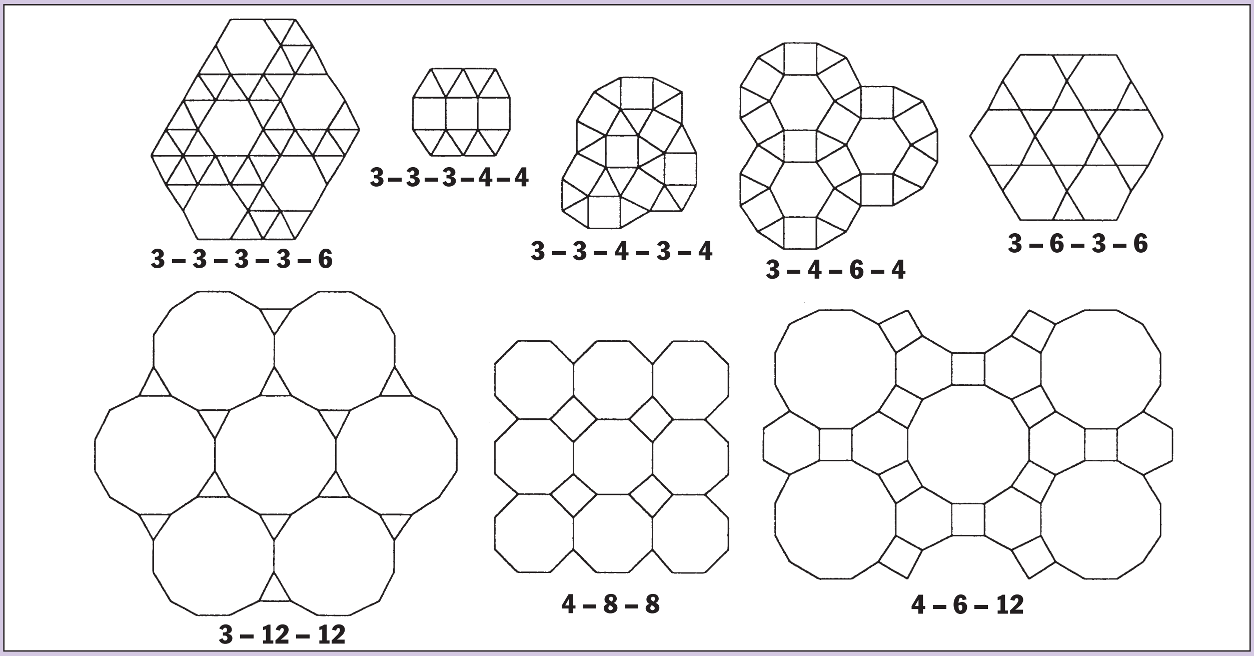


Fig. 7 **Arrangements of the eight semiregular tessellations**

Once the students understand semiregular tessellations, I have them create a full-page, multicolor tessellation. This activity should take between thirty minutes and two hours, depending on the level of investigation.

Angles and Regular Polyhedra

A REVIEW OF **FIGURE 1** WILL HELP ILLUSTRATE the connection between tessellations and polyhedra. In that activity, only three different regular polygons had interior angles that evenly divided 360 degrees. All other polygons either left a gap or overlapped when placed around a single point. For example, if two octagons are placed together, a gap is left that could be filled only by a square. Similarly, the gap left by placing two dodecagons together could be filled by a triangle to form another semiregular tessellation (see **fig. 8**). However, when three pentagons are placed together, the gap that remains has an angle of 36 degrees between the pentagons that cannot be filled by another regular polygon (see **fig. 8**).

The question that you can now pose to the students is "Since a regular polygon cannot be placed in the gap formed by three pentagons, can you think of another way to fill that gap?" Usually someone says, "Lift up the center point." By using a model with three pentagons taped together in a plane and a string taped to the center point, you can illustrate how lifting the center point can be done (see **fig. 9**). Remarkably, if you continue to build on these three pentagons with three pentagons at every point, the figure will close up completely, forming a regular polyhedron called a *dodecahedron* (see **fig. 10**). Because it consists of all the same type of polygon for faces and they all meet in the

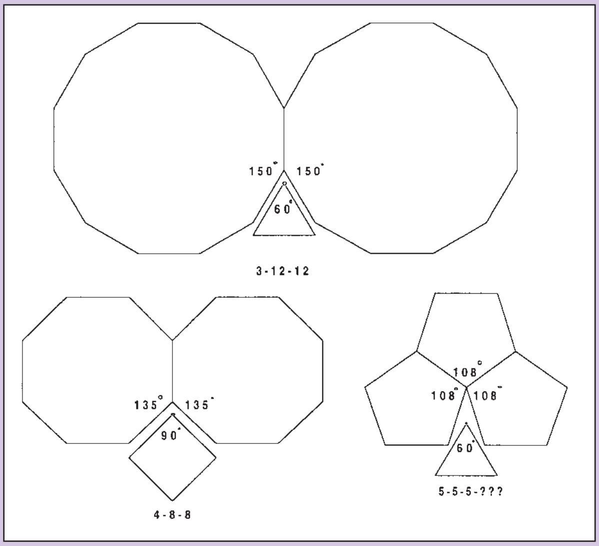


Fig. 8 **Gaps filled by other regular polygons**

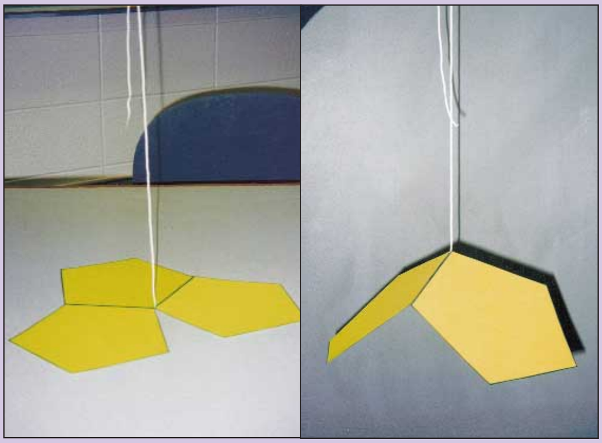


Fig. 9 **Lifting the center point**

same way at every point, it is considered a regular polyhedron. Because it has twelve faces, it has the specific name of dodecahedron, in which *do* means two, *deca* means ten, and *hedron* means plane or face. Regular polyhedra are also referred to as *Platonic solids*. By using this same type of argument with regular polygon arrangements, you can help students see why exactly five Platonic solids are possible.

Have the students put six triangles on their desks, arranged around a single point. Remove one triangle, and discuss the icosahedron. Remove two triangles, and discuss the octahedron. Remove three triangles, and discuss the tetrahedron.

Arranging three polygons that have more than five sides around a common point leaves no gap, and fewer than three will not form a three-dimensional object. Thus, only five regular polyhedra are possible (see **fig. 10**).

An interesting aspect of regular polyhedra is that when three or more polygons fit around a point and leave a gap, the corresponding three-dimensional figure closes up to form a regular polyhedron or a portion of one. In our discussion of semiregular tessellations, in some instances the polygons fit around a single point, but the pattern could not be repeated around every point to form a semiregular tessellation. With regular polyhedra, however, whenever three, four, or five of the same type of regular polygon fit around a single point, they fit around *all* points in the same way so that they eventually come together to form a polyhedron. The beauty of this investigation is that it relates to the first activity that the students did with the polygons in trying to fill the gaps, thus helping them make more connections.

More Polyhedra

AS WITH TESSELLATIONS, AN INTERESTING PURsuit is to examine an instance when more than one type of polygon meets at each point. The guiding principle is that at least three polygons are required and that the sum of the interior angles of the polygons around the point be less than 360 degrees. The gap in the plane will disappear when the model is lifted and becomes three-dimensional. Polyhedra with the same regular polygons meeting in the same arrangement around every point are called *semiregular* polyhedra. Thirteen semiregular polyhedra exist and are also referred to as *Archimedean solids*. A good discussion of these semiregular polyhedra appears in *Geometry: An Investigative Approach* (O'Daffer and Clemens 1992).

To construct one of the Archimedean solids, see **figure 11** and **table 2.** Students can select a solid and determine how many of each type of polygon are needed to construct it. The construction takes about an hour. Working in cooperative groups will be useful, since the construction is difficult for one pair of hands.

The final activity is building some big polyhedra, an interesting way to introduce the concept of similarity. Place students in groups of five or six, and have each group select a semiregular polyhedron that they would like to construct. It is helpful to

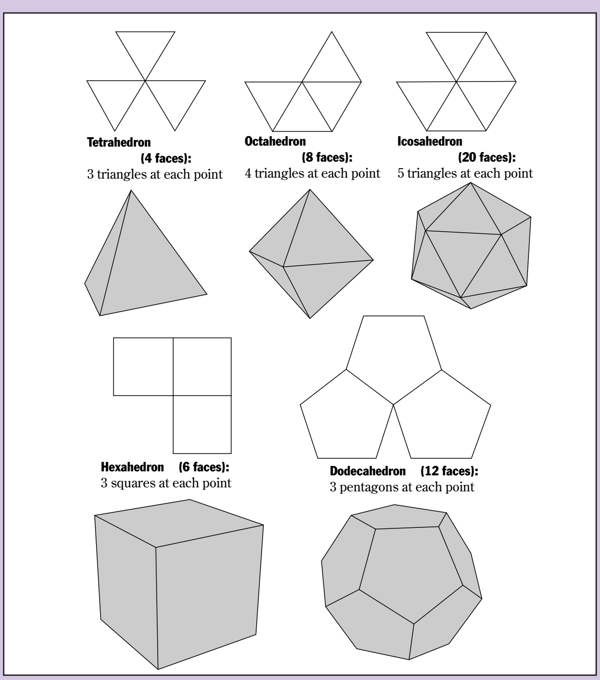


Fig. 10 The five Platonic solids

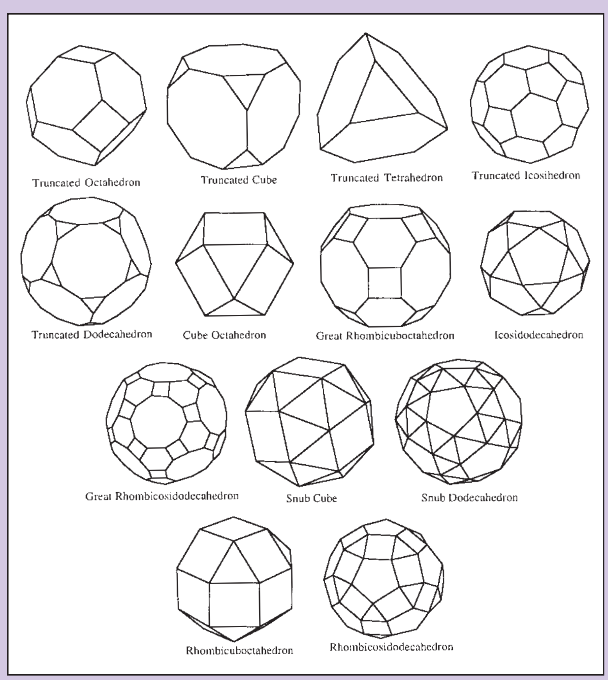


Fig. 11 The thirteen Archimedean solids

TABLE 2
List of Polygons Needed to Construct Any of the Thirteen Archimedean Solids

Polyhedron	Triangles	Squares	Pentagons	Hexagons	Octagons	Decagons
Truncated tetrahedron	4			4		
Truncated cube	8				6	
Truncated octahedron		6		8		
Truncated dodecahedron	20					12
Truncated icosahedron			12	20		
Cube octahedron	8	6				
Rhombicuboctahedron	8	18				
Great rhombicuboctahedron		12		8	6	
Snub cube	32	6				
Icosidodecahedron	20		12			
Rhombicosidodecahedron	20	30	12			
Great rhombicosidodecahedron		30		20		12
Snub dodecahedron	80		12			

Adapted from O'Daffer and Clemens (1992)

have a small version of the polyhedron as a model for constructing the large one. Using their small polygons as patterns, students can draw one of the polygons that they need to build their polyhedron, but on a larger scale; a side length might perhaps be seven, ten, or twelve inches. Remind students that the interior angles of a regular hexagon with side length one inch are the same as those of a regular hexagon with side length twelve inches. Repeat this process for each of the polygons that students need to construct the polyhedron that they have selected.

Cardboard works well for constructing polyhedra with long edges, but it can be a challenge to find a piece that is large enough to accommodate these large polygons. If locating cardboard is a problem, card stock will work, by placing one or two polygons per page.

Construct these very large polyhedra using two-inch-wide masking tape on the cardboard or regular tape on the card-stock version. The final product can be as large as six feet in diameter, depending on the type of polyhedra selected and the length of the sides of the polygons used. For further ideas on how to construct big polyhedra, see "Many Faces Have I" (Zilliox and Shannon 1997).

Summary

THE UNDERLYING THEME IN ALL THESE ACTIVITIES and explorations is the measure of the interior angles of regular polygons. By not having a specific algorithm for computing these angle measures, students are motivated to ask, "What are the angle measures?" and "Why are they important?" The resulting discussions lead them to see the connections among the angle measures, tessellations, and polyhedra, as well as real-world applications on and off the soccer field. From these connections, students gain mathematical understanding, which is, after all, our goal.

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One-Inch Stencils

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