Promote Systems of Linear Inequalities with Real-World Problems

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A key underlying principle of the NCTM's *Curriculum and Evaluation Standards* (1989) is the notion that instruction ought to arise out of problem situations. Moreover, since mathematics is a foundation for an ever-widening array of disciplines, "the curriculum for all students must provide opportunities to develop an understanding of mathematical models, structures, and simulations applicable to many disciplines" (NCTM 1989, 7).

Mathematical models, structures, and simulations are precisely the tools of operations research. The field of operations research is a rich source of real-world problem situations to which students can easily relate and within which mathematical concepts may be developed or skills practiced. For example, graphing systems of linear equations and inequalities, often without a meaningful practical context, has long been a staple of beginning high school algebra courses. At the same time, the graphs of such systems are typically used in operations-research textbooks to develop the concepts of linear programming, which are essential to understanding the solution of complex optimization problems (see, e.g., Winston [1994]).

The word experiment conjured up the picture of a scientist in a laboratory

We believe that bringing scaled-down real-world problem situations similar to those tackled by operations researchers into high school mathematics better motivates students to learn mathematics. Problems set in everyday situations have great potential for attracting and holding the attention of high school students because they deal with situations with which students have experience, for example, the clothes they wear, the places they work, or the lines in which they wait.

A BRIEF HISTORY OF OPERATIONS RESEARCH

Operations research has its roots in the years just before World War II, when the British prepared for the anticipated air war. In 1937, a new device, later called *radar,* was field-tested. The following summer, experiments began that explored how the information provided by radar could be used to direct the deployment and use of fighter planes. Until that time, the word *experiment* had conjured up the picture of a scientist carrying out a controlled experiment in a laboratory. In contrast, the multidisciplinary team of scientists working on this radar-fighter-plane project studied the actual operating conditions of these new devices and designed experiments in the field of operations. Thus was born the new term *operations research*. The team's goal was to understand the operations of the complete system of equipment, people, and such environmental conditions as weather or darkness and then improve on it. The work was an important factor in winning the Battle of Britain, and operations research eventually spread to all the military services.

In the United States, the first team working on antisubmarine tactics paralleled this approach. That group developed a series of mathematical models that they called *search theory,* which was

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used to develop optimal patterns of air search. Like their British counterparts, they got close to the action by riding in airplanes on patrol, just as the modern operations researcher might ride in a police car or spend time in an automotive-assembly plant. Every branch of the military currently has its own operations-research group that includes both military and civilian personnel. These groups play a key role in both long-term strategy and weapons development, as well as in directing the logistics of such actions as Operation Desert Storm. In addition, the National Security Agency has its own Center for Operations Research.

Operations research moved into the industrial domain in the early 1950s, and its growth paralleled the growth of the computer as a business-planning and management tool. As the field evolved, the core moved away from interdisciplinary teams to focus on developing mathematical models that can be used to model, improve, and even optimize real-world systems. These mathematical models include both deterministic models, such as mathematical programming, routing, or network flows; and probabilistic models, such as queuing, simulation, and decision trees.

MATHEMATICAL PROGRAMMING

The father of linear programming is George Dantzig, who developed its basic concepts between 1947 and 1949. During World War II, Dantzig worked on developing various plans, or proposed schedules, of training, logistical supply, and deployment, which the military calls *programs*. After the war, he was challenged to find an efficient way to develop these programs. Dantzig recognized that the planning problem could be formulated as a system of linear inequalities.

His next challenge involved the concept of a goal. When managers thought of goals at that time, they generally meant rules of thumb for carrying out a goal. For example, a naval officer might say, "Our goal is to win the war, and we can do that by building more battleships." Dantzig was the first to express the criterion for selecting a good or best plan as an explicit mathematical function, which we now call the *objective function*. All this work would have had limited practical value without an efficient method, or *algorithm*, for finding the best, or *optimal,* solution to a set of linear inequalities that maximizes an objective function, such as profit, or minimizes an objective function, such as cost. Dantzig developed the *simplex algorithm,* which efficiently solves this problem. Interestingly, in 1939 the Soviet mathematician and economist L. V. Kantorovich formulated and solved a linear-programming problem dealing with production planning. However, his work was essentially unknown even in the Soviet Union for twenty years and had no impact on the post–World War II development of linear-programming.

As mainframe computers grew more powerful, the first major users of the simplex algorithm to solve practical problems were the petroleum and chemical industries. One use was in minimizing the cost of blending gasoline to meet performance and content criteria. The field of linear programming grew rapidly and led to the development of nonlinear programming, in which inequalities or the objective function are nonlinear functions. In another extension of linear programming, called *integer programming,* some variables may take on only integral values. These techniques are known collectively as *mathematical programming*.

***Expressing the criterion for a best plan mathematically is called the* objective function**

In addition to the blending example, other applications of mathematical programming include scheduling workers to minimize labor costs, using a pattern or template that minimizes waste in cutting stock, and determining the production levels of different products to maximize a company's profit. In this article, we develop two problem situations of the last type, known as *product-mix problems*, through a series of student activities. Both examples were originally adapted for classroom use by an operations researcher. Although these examples are scaled-down versions of actual problems from industry, where such problems typically involve thousands of variables, they retain a real-world flavor.

MODELING A PRODUCT-MIX PROBLEM: THE LEGO FACTORY

Pendegraft (1997) suggests a way to use Lego plastic construction toys to model a product-mix problem situation for college students studying linear programming. A similar approach is easily accessible to high school algebra students.

The problem

Suppose that a factory manufactures only tables and chairs and that the profit on one chair is \$15 and on one table is \$20. Each chair requires one large piece of stock and two small pieces of stock, that is, one large Lego or other construction block and two small ones. Each table requires two large and two small pieces of stock. **Figure 1** shows a table and a chair. Finally, suppose that you have only six large and eight small pieces of stock. How many chairs and how many tables should you build to maximize profit?

A concrete exploration

The student activity works best if students actually have construction blocks such as Lego blocks with which to build "tables" and "chairs." However, if they are unavailable, other materials could be substituted, for example, two different sizes or colors of plas-

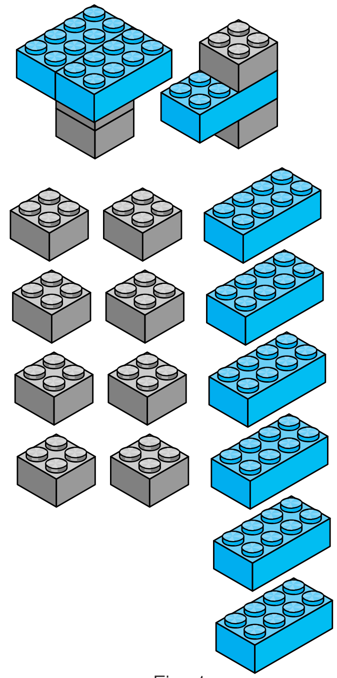

Fig. 1
A Lego table, chair, and available resources

Legos are used to model a product-mix problem

tic tiles or cubes. It is important to allow students an opportunity to see what they can build using the available materials and to determine the profit for each possibility. Later, during a more abstract exploration of this problem, the abstract concepts can easily be linked to this more concrete exploration.

Students might compete in pairs or in small groups to obtain the optimal solution. Students will quickly see that only four logical possibilities for the optimal solution exist—(1) four chairs, (2) three chairs and one table, (3) two chairs and two tables, and (4) three tables—and they should have no trouble determining the profit for each possibility. It will then be evident that building two chairs and two tables maximizes profit. **Figure 2** shows this solution.

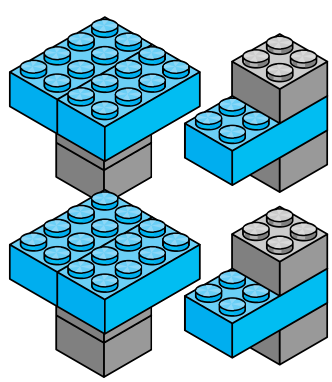

Fig. 2
Two of each is the optimal solution.

An abstract exploration

After students have completed their concrete exploration of the Lego problem, that exploration can be linked to a more abstract one that uses the terminology of mathematics and operations research. The steps in the abstract exploration involve—

- defining a set of *decision variables* that completely describe the decision to be made;
- modeling the problem's objective by using the decision variables to define an *objective function;*
- identifying any *constraints,* or restrictions, on the decision variables, such as the limited resources available;
- graphing the system of constraints to locate a *feasible region;* and
- determining which solution within the feasible region is the *optimal solution*.

On the basis of their concrete exploration of the Lego example, students should be able to identify two decision variables—

C = the number of chairs built

and

T = the number of tables built.

Students should next model the objective of the problem by defining the profit, P, as the objective function $P = 15C + 20T$. They should also identify two constraints on what they were able to build, because they were given only six large and eight small Lego blocks. Translating these constraints into inequalities using the decision variables may take some probing. Many students may try to write such constraints as—

1 table = 2 large pieces + 2 small pieces

or

1 chair = 1 large piece + 2 small pieces

because that is how each item is constructed. However, the constraints concern the consumption of limited resources, so students will need to focus separately on the number of large and small Lego pieces and how each of these resources is consumed in constructing a table or a chair. Since a chair requires one large piece and a table requires two large pieces, students should find that $1C + 2T \leq 6$, since a chair and a table each require two small pieces, $2C + 2T \leq 8$.

After students identify the system of constraints, they should graph the system to locate the feasible region. Because students often do not view graphs in the same ways that teachers do (Dunham and Osborne 1991), you may want to include some activities to help students decide which half-plane to include in the graph of each inequality. For example, students might test representative points to see whether they satisfy the inequality. **Figure 3** shows a graph of the feasible region with the four logical possibilities for the optimal solution labeled with their coordinates. Students should be asked to interpret each of those points as they relate to the problem situation, so that they begin to link the graph with their previous exploration.

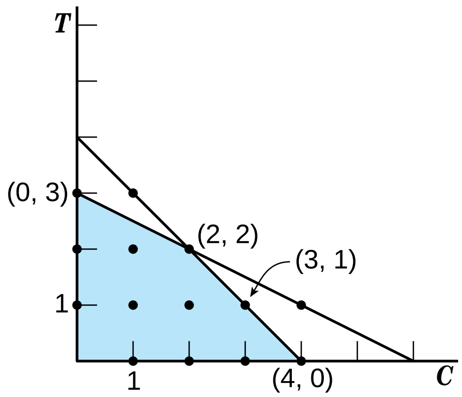

Fig. 3
The feasible region

You may next want to discuss with the students the discrete nature of the feasible region for this example. Since the decision variables must take on integral values, the feasible region actually consists only of the lattice points in the shaded region. In real-world applications, these problems are usually formulated in terms of hourly or weekly production rates, and continuous variables are acceptable. The feasible region is then the entire shaded region.

You should also determine whether students understand why the possibility of building two chairs and two tables renders building one chair and two tables nonoptimal. When students comprehend this concept, the teacher can ask whether they notice anything about the location in the feasible region of the four points that were logical possibilities for the optimal solution. The location of these points on the boundary of the feasible region will play an important role in the next student exploration.

SOLVING A PRODUCT-MIX PROBLEM: THE HIGH STEP SHOE CORPORATION

The example that follows presents students with a somewhat more abstract exploration of a product-mix problem situation. This example develops the principle that a unique optimal solution to a linear-programming problem always occurs at a *corner point* on the boundary of the feasible region.

The problem

Mr. M. Jordan, director of manufacturing for the High Step Sports Shoe Corporation, wants to maximize the company's profits. High Step makes two brands of sport shoes, Airheads and Groundeds. Each pair of Airheads returns a \$10 profit, and each pair of Groundeds returns a profit of \$8.50.

The steps in manufacturing the shoes include cutting the materials on a machine and having workers assemble the pieces into shoes. Six machines cut the materials, 850 workers assemble the shoes, and the factory operates forty hours per week.

Each cutting machine can actually perform only fifty minutes of work in an hour because of the time required to calibrate and maintain the machines. Each pair of Airheads requires three minutes of cutting time, whereas each pair of Groundeds requires two minutes of cutting time. A worker takes an average of seven hours to assemble a pair of Airheads and an average of eight hours to assemble a pair of Groundeds.

Mr. Jordan's goal is to maximize High Step's profits, subject to all those constraints.

The student exploration

After students have read and discussed the problem situation, they should identify the decision variables and define an objective function. In this problem, the decision variables are—

A = the number of pairs of Airheads manufactured each week

and

G = the number of pairs of Groundeds manufactured each week.

The objective function is then profit, $P = 10A + 8.5G$.

A lively discussion of slope should ensue

Next, students must explore the constraining conditions. First, one constraint is related to the total machine time available per week. Since each machine can perform only fifty minutes of work each hour, the six machines working together can provide only

$$6 \times 50 \text{ min/h} \times 40 \text{ h/wk.} = 12\,000 \text{ min/wk.}$$

of cutting time. We know that each pair of Airheads requires three minutes of cutting time and each pair of Groundeds requires two minutes of cutting time, so

$$3A + 2G \leq 12\,000.$$

A second constraint is related to the total worker time available each week for assembly. The 850 assembly workers provide

$$850 \times 40 = 34\,000 \text{ h/wk.}$$

of assembly time. Again, we know that seven hours are required to assemble each pair of Airheads and eight hours to assemble each pair of Groundeds. Thus,

$$7A + 8G \leq 34\ 000.$$

Some students may not see why it is desirable to express the machine and assembly times available on a weekly basis. The class could discuss the arbitrary nature of this aspect of the problem formulation. These rates, as well as the decision variables, could just have well been expressed on a daily or hourly basis. It is important only to express these rates consistently.

Some students may also be bothered because the machine constraint is expressed in minutes per week, whereas the assembly constraint is expressed in hours per week. You may want students to reformulate the assembly constraint using minutes per week as the basic unit. Converting 7, 8, and 34 000 hours to minutes yields

$$420A + 480G \leq 2\ 040\ 000.$$

But then factoring 60 out of each term in the last inequality and dividing both sides by 60 leaves the original assembly constraint intact:

$$7A + 8G \leq 34\ 000$$

Two constraints remain that are not explicitly stated in the description of the problem. Since A and G both represent a number of pairs of shoes manufactured in a week, each variable is nonnegative. Thus, we have $A \geq 0$ and $G \geq 0$. You may want to ask students whether either or both of these variables could be zero.

The system of constraints defines the feasible region, which is shown in **figure 4**. The magnitude of appropriate values for A and G makes the scale of the graph an important consideration. Because the exploration of optimality that follows depends on an accurate graph, you will want to be certain that each student or group of students has such a graph before proceeding to the optimal solution. It is also helpful for students to find the coordinates of the vertices of the feasible region at this time, as those points will be important in developing the optimal solution.

Such a problem cannot be solved with paper and pencil

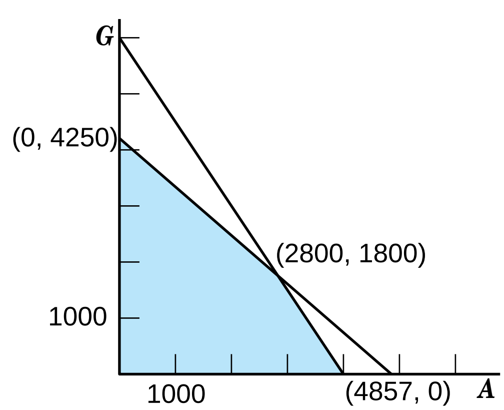

Fig. 4
High Step Shoes feasible region

The best solution for this problem gives the High Step Sports Shoe Company a maximum profit. To determine this optimal solution, a number of strategies could be used. However, if the problem has a unique optimal solution, that solution must lie at one of the corner points of the feasible region. This principle is known as the *corner principle*.

One way to help students understand the corner principle is to identify a number of points in the feasible region then use the objective function to compute the profits for each point selected. Small groups of students could be assigned the task of finding in the feasible region the point that generates the maximum profit. After students have computed the profit, P_i, for a number of points in the feasible region, they should graph an equation $10A + 8.5G = P_i$ corresponding to each point. Students will then need only to observe that $10A + 8.5G = P_i$ defines a family of parallel lines and that the further right and higher the line lies in the coordinate system, the greater the profit is. Thus, the line that is highest and farthest right while still intersecting the feasible region always passes through a corner point, and this point corresponds to the optimal solution.

Figure 5 shows the feasible region of the High Step Shoe problem and a family of parallel lines, including the line passing through (2800, 1800), the optimal solution.

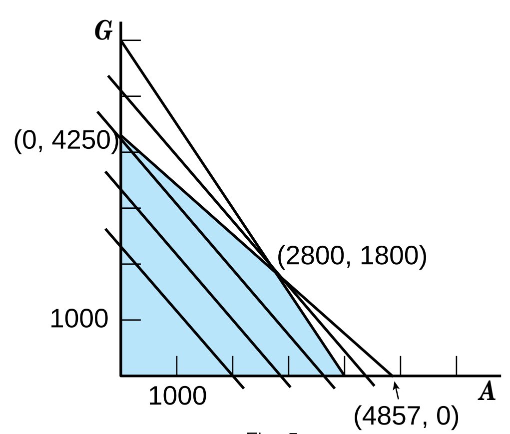

Fig. 5
A family of parallel lines

When the family of parallel lines defined by the objective function is actually parallel to the boundary of the feasible region, the optimal solution may no longer be unique. Any point lying on that portion of the boundary is optimal. In an example where the decision variables must take on integral values, as in the High Step example, more than one point along that boundary could have integral coefficients. The optimal solution then is not unique. If the nature of the problem does not restrict the decision variables to integral values, then any point along that portion of the boundary parallel to the family of lines defined by the objective function is optimal.

A PRODUCT-MIX PROBLEM FROM THE REAL WORLD

Brown, Graves, and Honczarenko (1987) describe the use of a linear-programming model to solve a real problem for Nabisco. Production in the Biscuit Division of Nabisco involves baking and such secondary operations as sorting, packaging, and labeling.

Scheduling and operating bakeries are difficult tasks. Each oven can produce some, but not all, of the products. The efficiency of the ovens varies. Several ovens can simultaneously share the facilities for secondary operations at one site. In addition, production must be planned to keep manufacturing and transportation costs at a minimum.

Some questions addressed by the mathematical model at Nabisco include the following:

- Where should each product be produced?
- How much of each product should be assigned to each oven?
- From where should each product be shipped to each customer?

A realistic problem at Nabisco could involve 150 products, 218 facilities, 10 plants, and 127 customer zones. A problem of this size involves more than 40 000 decision variables and almost 20 000 constraints.

Obviously, a problem of this magnitude cannot be solved using a paper-and-pencil graphical approach. However, the same sort of problem-formulation skills that are developed in such examples as the Lego factory or High Step Shoes are used to construct mathematical models of such problems as Nabisco's. After the problem has been mathematically formulated, it can be solved routinely using a computer with appropriate software. Indeed, when the Nabisco problem was solved in 1983 on an IBM 3033 computer, less than sixty seconds of CPU time was required. Today, the same problem is solvable on a microcomputer.

CONCLUSION

Mathematical programming is just one in an array of mathematically based techniques that are used in the field of operations research and are accessible to high school students. Other techniques include routing, queuing, logistics, and simulation. Because operations researchers solve problems in the real world, operations-research-based problems have rich connections to the world in which students live and work. Drawing on such problem situations is one way in which teachers can let applications of mathematics drive instruction. We believe that doing so will better motivate students to learn the mathematics they encounter in the classroom.

Teachers can let applications of mathematics drive instruction

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