Example 13.6

Probabilistic Inventory Models

Background Information

- In Example 13.1 we considered the Machey's department store, which sells, on average, 1200 cameras per year.
- The store pays a setup cost of $35 per order, and the holding cost is $10 per camera per year.
- It takes 1 week for an order to arrive once it is placed.
- In that example the optimal order quantity *Q* was found to be 92 camera. Now we assume that the annual demand is normally distributed with mean 1200 and standard deviation 70.
- Machey's wants to know when to order and how many cameras to order at each ordering opportunity.

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Solution

- Suppose the company places an order for *Q* cameras every time its inventory level drops to *R*.
- Our goal is to find “good” values of *Q* and *R*.
- There are two aspects of this model that are critical to its solution: demand during lead time and the “cost” of running out of inventory.

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Demand During Lead Time and Safety Stock

- The most critical probabilistic quantity is the amount of demand during an order lead time.
- To illustrate, suppose that Machey’s uses *R*=30 as the reorder point.
- This means that an order is placed as soon as the inventory level drops to 30 cameras. This order will arrive 1 week later.
- The demand during lead time, in conjunction with the choice of *R*, determines the extent of shortages. Before we can continue, we need to analyze this quantity in some detail.

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Demand During Lead Time and Safety Stock -- continued

- The Excel function NORMDIST can be used.
- The syntax for this function is NORMDIST(x,m,s, 1).
- It returns the probability of a normal random variable with mean m and standard deviation s being to the left of the specified value x.
- Therefore, we find $P(D\text{LD} > 33)$, the probability of a stockout, with the formula **=1-NORMDIST(33,23,9.7,1)**, which is approximately 0.15.

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Demand During Lead Time and Safety Stock -- continued

- In general, suppose that Machey's decides to set R equal to k standard deviations above the mean, where k is a multiplier that must be determined. That is, it uses the reorder level

  $$R = m_{LD} + k s_{LD} = m_{LD} + \text{safety stock}$$

- In effect, the multiplier k becomes the decision variable. Usually k is positive. The term $k\sigma_{LD}$ above μ_{LD}.

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Demand During Lead Time and Safety Stock -- continued

- Therefore, it expects the inventory level to be $k\sigma_{LD}$, a positive value, when the order arrives.
- This value is its cushion against larger than expected demand – hence the term “safety stock”.
- But although the company plans for this safety stock to exist, there is no guarantee that it will exist.

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Finding the Expected Costs

- As we saw in the previous probability calculation *k*=1, there is about a 15% chance that the safety stock of 10 units will be depleted before the order arrives and a stockout will occur.
- We want to choose *k* and the order quantity in an optimal manner.
- We now develop an expression for Machey’s expected total annual cost as a function of the order quantity *Q* and *k*.

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Finding the Expected Costs -- continued

- In the following discussion we will refer to **an order cycle**.
- Such a cycle begins each time an order arrives and ends just before the next order arrives.
- We first consider the annual setup and holding costs.
- The expected annual setup cost is Km_{AD}/Q and the expected annual holding cost is $h(Q/2+ks_{LD})$ where K=\$35, h=\$10, m_{AD}=1200, s_{LD}= 9.7, and Q and k need to be determined.

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Two ways to "Cost" Shortages

- We now consider two alternative models of "costing" shortages.
- It is important to realize that neither of these models is clearly superior to the other.
- Model 1 assumes that there is a shortage cost of p per unit short, In this model, a cycle with a shortage of 5 units is 5 times as costly as a cycle with a shortage of only 1 unit

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Two ways to “Cost” Shortages -- continued

- Model 2 gets around the difficult problem of assessing dollar shortage costs by instead specifying a “service level”.
- Specifically, it requires that the fraction of demand that can be met from on-hand inventory must be at least *s*, where *s* is a number between 0 and 1.
- Before we can solve Machey’s problem on a spreadsheet, we must develop formulas for the shortage cost for these two shortage-costing models.

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Expected Shortage Cost for Model 1

- In model 1 Machey’s assesses a shortage cost of *p* per unit short during any order cycle.
- Therefore, to evaluate the expected annual shortage cost, we must find the expected number of shortages per order cycle.
- Let *E*(*B*) be the expected number of units short during a typical order cycle.
- Then the expected shortage cost during this cycle is *pE*(*B*), and the expected annual shortage cost is the expected shortage cost per cycle multiplied by the expected number of cycles per year m_{AD}/Q.

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Expected Shortage Cost for Model 1 -- continued

- This leads to the following expected total annual cost:
  Model 1 expected annual shortage cost = $pE(B)m_{AD}/Q$
- The problem is to find an expression for *E(B)*. It can be shown that this expected value is related to a well-known quantity called the **normal loss function**.
- Fortunately this can be calculated with built-in Excel functions.

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CAMERA1.XLS

- We now show how to implement model 1 for the camera example. We assume Machey's decides to use model 1, with *p*=$10 as the shortage cost.
- The spreadsheet model appears on the next slide.
- This file contains the model.

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	A	B	C	D	E	F
1	**Optimal (R,Q) inventory policy for model 1**					
2						
3	**Inputs**					
4	Setup cost per order	$35				
5	Holding cost per unit per year	$10				
6	Expected annual demand	1200				
7	StDev of annual demand	70				
8	Lead time in years	0.0192				
9	Expected demand during lead time	23.077				
10	StDev of demand during lead time	9.707				
11	Shortage cost incurred each cycle with a shortage	$100				
12						
13	**Optimal solution using Solver to choose Q and k**					
14	Order quantity	96.2				
15	Factor k for safety stock calculation	1.81				
16	Safety stock	17.6				
17	Probability of a shortage during an ordering cycle	0.04				
18	Reorder point	40.6				
19						
20	Annual setup cost	$437				
21	Annual holding cost	$656				
22	Annual shortage cost	$44				
23	Total annual cost	$1,137				

Range names used:
SetupCost - B4
UnitHoldCost - B5
ExpAnnDemand - B6
StdevAnnDemand - B7
LeadTime - B8
ExpDDLT - B9
StdevDDLT - B10
UnitShortCost - B11
OrderQuan - B14
Factor_k - B15
SafetyStock - B16
PrShort - B17
TotAnnCost - B23

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Developing Model 1

- The model can be formed as follows.
  - **Inputs**. Enter the inputs in the shaded range.
  - **Lead time demand**. Calculate the mean and standard deviation of lead time demand in cells B12 and B13 with the formulas **=LT*MeanAD** and **=SQRT(LT)*StdevAD**.
  - **Decision variables**. Enter any values in cells B16 and B17 for the order quantity *Q* and the multiplier *k*. These will be the changing cells.

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Developing Model 1 -- continued

– **Safety stock and reorder point**. The decision variables determine the safety stock and reorder point. Calculate them in cells B18 and B19 with the formulas **=k*StDevLD** and **=MeanLD+SS**

– **Expected backorders**. Use the equation to calculate the expected number of backorders per order per order cycle, *E(B)*, in cell B20 with the formula **=(NORMDIST(k,0,1,0)-k*(1-NORMSDIST(k)))*StdevLD**

Note that this formula uses two related functions, NORMDIST and NORMSDIST. The first of these takes four arguments: a value, the mean, the standard deviation, and 0 or 1. When the fourth argument is 1, the function returns a cumulative probability, but when this argument is 0, it returns the value of the density function.

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Developing Model 1 -- continued

– **Expected annual costs**. Use the equations to calculate the expect annual setup, holding, and shortage costs in cells B22-B24 with the formulas **=SetupCost*MeanAD/OrderQuan**, **=HoldCost*(SS+OrderQuan/2)** and **=ShortCost*MeanShort*MeanAD/OrderQuan**. Then calculate the expected total annual cost in cell B25 by summing the costs in cells B22-B24.

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Using the Solver

- We setup the Solver to minimize the expected total annual cost.
- The only constraints are nonnegativity constraints on the changing cells, B16 and B17.
- As usual, we do not check the Assume Linear Model box. This model is nonlinear in both Q and k.
- The interpretation of the Solver solution shown in the model is that Machey's should wait until the inventory level drops to approximately 37 cameras and then place an order for 96 cameras.

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Using the Solver -- continued

- The actual number of backorders during an order lead time is a discrete random variable with possible values 0, 1, 2, and so on.
- However, the expected number of backorders, a weighted average of these possible values is $E(B)$ = 0.35, so that the expected shortage cost during any order cycle is $pE(B)$=$3.50.
- Multiplying this by the expected number of cycles per year (1200/96.2) gives the expected annual shortage cost of approximately $44.

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Service Level Constraint for Model 2

- Model 2 uses a service level constraint instead of a dollar shortage cost.
- To model this constraint, we need an expression for the fraction of demand met directly from existing inventory.
- Note that *Q* items are ordered each cycle, and the expected shortage per cycle is *E(B)*, which we evaluated for model 1.
- Therefore, the expected fraction of demand met on time is $1 - E(B)/Q$, and the model 2 service level constraint becomes: $1 - E(B)/Q \geq s_2$.

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CAMERA2.XLS

- We now show how to implement model 2 for the camera example. It assumes a service level where at least 98% of customer demands must be satisfied with existing inventory.
- The spreadsheet model appears on the next slide.
- This file contains the model.

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	A	B	C	D	E	F	G
1	**Optimal (R,Q) inventory policy for model 3**						
2							
3	**Inputs**						
4	Setup cost per order	$35					
5	Holding cost per unit per year	$10					
6	Expected annual demand	1200					
7	StDev of annual demand	70					
8	Lead time in years	0.0192					
9	Expected demand during lead time	23.077					
10	StDev of demand during lead time	9.707					
11							
12	**Optimal solution using Solver to choose Q and k**						
13	Order quantity	91.7					
14	Factor k for safety stock calculation	1.64					
15	Safety stock	16.0					
16	Fraction of cycles with no shortages	0.950	>=	0.95			
17	Fraction of all demand met on time	0.998					
18	Reorder point	39.0					
19							
20	Annual setup cost	$458					
21	Annual holding cost	$618					
22	Total annual cost	$1,076					

Range names used:
SetupCost - B4
UnitHoldCost - B5
ExpAnnDemand - B6
StdevAnnDemand - B7
LeadTime - B8
ExpDDLT - B9
StdevDDLT - B10
OrderQuan - B13
Factor_k - B14
SafetyStock - B15
PrNoShort - B16
ReqdPrNoShort - D16
TotAnnCost - B22

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Developing Model 2

- This model is similar to the one shown for Model 1, so we list only the changes.
  - **Required service level**. There is no unit shortage cost input. Instead, enter the required service level in cell D22.
  - **Actual service level**. Use the left side of the inequality to calculate the expected fraction of demand met with existing inventory in cell B22 with the formula **=1-MeanShort/OrderQuan**
  - **Expected total annual cost**. The total cost now includes only the setup and holding costs (which are the same as before).

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Using the Solver

- We again minimize the expected total annual cost, but now add the service level constraint in row 22.
- There is no longer a shortage cost to penalize shortages.
- Instead the company simply requires that 98% of all demand be met out of existing inventory.
- Compared to the solution for model 1, the solution shown for model 2 has a slightly larger order quantity *Q* and a significantly lower multiplier *k*.

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Using the Solver -- continued

- Therefore, this model specifies that Machey's should order a bit more on each order, and it should hold less safety stock – that is, it should let its inventory drop lower before ordering.

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Comparing the Models

- Why are the solutions from the two models different?
- One way to understand the difference is to substitute the optimal values of *Q* and *k* from model 1 into the spreadsheet for model 2.
- If you do this, you will find that *Q* and *k* from model 1 lead to a service level of 0.996 in model 2.
- This larger service level can be attained only with increased safety stock.

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Comparing the Models -- continued

- Evidently, the unit penalty cost of $10 in model 1 is equivalent to a required service level of 0.996 in model 2.
- Alternatively, if we want a service level of 0.98 in model 2, then the equivalent model 1 unit penalty cost must be considerably less than $10.
- Machey's might favor model 2 because a service level constraint is easier to estimate than a unit shortage cost. However, any particular service level in model 2 is really equivalent to an appropriate unit shortage cost in model 1.

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Comparing the Models -- continued

- To find the equivalent unit shortage cost *p* for any required service level *s*, we could proceed as follows.
- First, we would add a formula in the model 1 spreadsheet to capture the expected fraction of demand met with existing inventory.
- Then we would run the Solver repeatedly on model 1, each time with a different value of *p*, until the service level is equal to *s*.
- The results are shown on the next slide.

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Comparing the Models -- continued

Equivalent Shortage Costs and Service Levels for Camera Example

p_2	Service Level
$10	99.6%
$8	99.5%
$6	99.3%
$4.50	99.0%
$2.60	98.0%

- This shows, for example, that a required service level of 98% is evidently not very stringent.
- It is equivalent to a unit shortage cost of only $2.60.

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Random Lead Times

- We have assumed that the lead time for orders is a known quantity.
- It is not difficult to modify the analysis for the case where the lead time *L* is random.
- This is important, because it is not at all uncommon in real applications for ordering lead times to be uncertain – suppliers might not be able to deliver according to a precise schedule.

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Random Lead Times -- continued

- When *L* is random, we need to estimate its mean and standard deviation, which we denote by μ_L and σ_L. Given these values the expected demand during lead time becomes $\mu_{LD} = \mu_L \mu_{AD}$ and the standard deviation of demand during lead time becomes

$$\sigma_{LD} = \sqrt{\mu_L \sigma_{AD}^2 + \mu_{AD}^2 \sigma_L^2}$$

- Completing the calculation, we see that the extra uncertainty about the lead time adds to the uncertainty about the demand during lead time.

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Random Lead Times -- continued

- Once we use these formulas to obtain μ_{LD} and σ_{LD}, we find the optimal (R,Q) exactly as in the nonrandom lead time case.
- When the lead time is uncertain, a company needs to order earlier, which means larger safety stock and higher inventory holding costs.

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