

Review of Differential Equations

Dr. Robinson

First Order Differential Equations

Definitions and the existence/uniqueness theorem

Definition 1 A *first order differential equation* for the unknown function $y = y(t)$ is an equation of the form

$$\frac{dy}{dt} = f(t, y) \quad (1)$$

where $f(t, y)$ is a given function of the variables t and y .

Definition 2 An *explicit solution* of equation (1) is any function $y(t)$ that identically satisfies the equation (typically on some interval $a < t < b$). An *implicit solution* is a solution defined by an equation $F(t, y) = 0$.

Theorem 3 If f and $\frac{\partial f}{\partial y}$ are continuous on a rectangular region containing the point (t_0, y_0) , then there exists a unique solution of the *initial value problem*

$$\frac{dy}{dt} = f(t, y) \quad (2)$$

$$y(t_0) = y_0 \quad (3)$$

valid for t in some interval $(t_0 - \delta, t_0 + \delta)$.

Methods of solution

Separable equations

If $f(t, y) = g(t)h(y)$, then

$$\frac{1}{h(y)} \frac{dy}{dt} = g(t) \Rightarrow \int \frac{1}{h(y)} dy = \int g(t) dt$$

so the solution of (1) is reduced to finding two antiderivatives. Note that every **autonomous equation** ($f(t, y) = f(y)$) is separable.

Linear equations

If $f(t, y) = -p(t)y + q(t)$, then

$$\frac{dy}{dt} + p(t)y = q(t) \Rightarrow \frac{d}{dt}(\mu(t)y) = \mu(t)q(t)$$

where $\mu(t)$ is the **integrating factor** $\mu(t) = e^{\int p(t) dt}$. The equation on the right above is then solved by integrating both sides.

For linear initial value problems the Existence and Uniqueness Theorem can be strengthened to guarantee existence and uniqueness throughout the largest interval (containing the initial time t_0) for which $p(t)$ and $q(t)$ are continuous.

Homogeneous equations

If $f(t, y) = f(y/t)$, then the substitution $z(t) = \frac{y(t)}{t}$ results in a separable equation for z .

Bernoulli equations

If $f(t, y) = -p(t)y + q(t)y^n$, then the substitution $z(t) = y(t)^{1-n}$ results in a linear equation for z .

Exact equations

If the equation can be written in the form (perhaps using an integrating factor) $M(t, y) + N(t, y) \frac{dy}{dt}$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ then the equation is **exact** and the solution is given by $F(t, y) = c$ where $\frac{\partial F}{\partial t} = M$ and $\frac{\partial F}{\partial y} = N$.

Numerical solution

Euler's method

Euler's method uses the tangent line approximation

$$y(t) \approx y(t_0) + y'(t_0)(t - t_0)$$

to produce the approximate solution to the initial value problem (2-3)

$$y(t_n) \approx y_n = y_{n-1} + h f(t_{n-1}, y_{n-1})$$

for $n = 1, 2, \dots$ where $t_n = t_0 + nh$. The global error in the approximation is $O(h)$.

Second Order Differential Equations

Definitions and the existence/uniqueness theorem for linear equations

Definition 4 A *second order differential equation* for the unknown function $y(t)$ is an equation that can be written in the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) \quad (4)$$

where f is a given function of the three variables t , y , and $\frac{dy}{dt}$.

If the function f in (4) is independent of the variable y , the change of variables $v = \frac{dy}{dt}$ results in a first order equation for v . If this equation can be solved, then y can be obtained by antidifferentiating v . If f is independent of t (i.e., equation (4) is **autonomous**), then the change of variables $v = \frac{dy}{dt}$, but with v viewed now as a function of y , results in a first order equation for $v(y)$. If this equation can be solved, then y can be obtained as the solution of the first order separable equation $\frac{dy}{dt} = v(y)$.

Definition 5 A *linear second order differential equation* for the unknown function $y(t)$ is an equation that can be written in the form

$$\frac{d^2y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = R(t) \quad (5)$$

where P , Q , and R are given functions of the variable t .

Theorem 6 If P , Q , and R are continuous on an open interval I containing t_0 , then there exists a unique solution of the **initial value problem** consisting of equation (5) with initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0^*$. This solution is valid for all t in the interval I .

Homogeneous equations

Equation (5) is **homogeneous** if the right side is identically zero, i.e., if

$$\frac{d^2y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0. \quad (6)$$

The **general solution** of the homogeneous solution can be expressed as a linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for any two **linearly independent** ($y_1 y_2' - y_1' y_2 \neq 0$) solutions $y_1(t)$ and $y_2(t)$.

Equations with constant coefficients

Equation (6) has **constant coefficients** if P and Q are constant. The general solution of the homogeneous equation

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0$$

is found by first solving the associated quadratic equation

$$ar^2 + br + c = 0.$$

If $b^2 - 4ac > 0$, there are two real roots r_1, r_2 and

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If $b^2 - 4ac = 0$, there is a single real root r and

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

If $b^2 - 4ac < 0$, there are two complex roots $\alpha \pm i\beta$ and

$$y(t) = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t).$$

Cauchy-Euler equations

Equation (6) is a (homogeneous) **Cauchy-Euler** equation if it can be put in the form

$$at^2 \frac{d^2 y}{dt^2} + bt \frac{dy}{dt} + cy = 0$$

with a, b , and c constant. The substitution $t = e^x$ (or, $t = -e^x$) gives a second order linear equation with constant coefficients for $y(x)$.

Other equations

There is no general solution procedure for second order linear homogeneous equations with non-constant coefficients. If one solution, say $y_1(t)$, is known (say, by means of a lucky guess), a second solution $y_2(t)$ can be found using **reduction of order**. When all else fails, try looking for solutions in the form of a **series**.

Reduction of order The substitution $y(t) = z(t)y_1(t)$, when $y_1(t)$ is a known solution, gives a first order linear equation for $z'(t)$. Solving this equation gives the second linearly independent solution $y_2(t) = z(t)y_1(t)$.

Series solutions Substituting

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

into (5) and, if necessary, expanding $P(t)$ and $Q(t)$ in similar series, one can try to find a recursion for the coefficients a_n . If initial conditions are given, then a_n for $n \geq 2$ can be computed using $a_0 = y(t_0)$ and $a_1 = y'(t_0)$. If $P(t)$ or $Q(t)$ do not have series expansions at $t = t_0$ the one may resort to the **method of Frobenius** which involves seeking a solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^{n+\lambda}$$

where $a_0 \neq 0$ and λ is to be determined.

Nonhomogeneous equations

The general solution of the nonhomogeneous equation (5) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

where y_1 and y_2 are linearly independent solutions of the homogeneous equation ($R(t) \equiv 0$) and y_p is *any* solution of the nonhomogeneous equation. One can use the **method of undetermined coefficients** (judicious guessing) to find y_p or resort to the method of **variation of parameters**.

Undetermined coefficients ("Judicious Guessing")

This method relies on guessing the form of the particular solution based on the form of the equation and, particularly, the right side $R(t)$. It is most convenient for equations with constant coefficients where the right side $R(t)$ is (most generally) a sum of products of the form

$$R(t) = p(t)e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$$

where $p(t)$ is a polynomial in t . The assumed form for y_p is typically of the same form as $R(t)$, but this form must be multiplied by t or t^2 in the instances where $R(t)$ contains a solution of the corresponding homogeneous equation.

Variation of parameters

Once the solutions y_1 and y_2 of the homogeneous equation are known, a particular solution is given by

$$y_p(t) = -y_1(t) \int \frac{y_2(t)R(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t)R(t)}{W(t)} dt$$

where $W(t)$ denotes the **Wronskian** $W = y_1y_2' - y_1'y_2$.

Laplace transforms

Linear equations are sometimes solvable using the Laplace transform

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st} dt$$

where $s > 0$ and the inverse transform

$$f(t) = \mathcal{L}^{-1}(F(s)).$$

The strategy is to take the transform of both sides of the equation, use relations such as

$$\begin{aligned} \mathcal{L}(y') &= s\mathcal{L}(y) - y(0), \\ \mathcal{L}(y'') &= s^2\mathcal{L}(y) - sy(0) - y'(0), \end{aligned}$$

to obtain an equation which can be solved for $\mathcal{L}(y)$, and then recover $y(t)$ using the inverse transform.

Systems of First Order Equations

Definitions and the existence/uniqueness theorem

Definition 7 A *system of first order differential equations* for the unknown functions $x(t)$ and $y(t)$ has the form

$$\frac{dx}{dt} = f(x, y, t) \tag{7}$$

$$\frac{dy}{dt} = g(x, y, t) \tag{8}$$

where f and g are known functions. If f and g are both linear functions of x and y , then the system is said to be **linear**.

Theorem 8 If f , g , and their first order partial derivatives with respect to x and y are all continuous for (x, y, t) in some box-shaped region containing (x_0, y_0, t_0) then the system (7-8) has a unique solution satisfying $x(t_0) = x_0$, $y(t_0) = y_0$ and valid in some interval $t_0 - \delta < t < t_0 + \delta$. The parametric curve $(x(t), y(t))$ is called an **orbit**.

Second order equations as systems of first order equations

Any second order differential equation

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}, t\right)$$

can be converted to a system of first order equations by defining $y = \frac{dx}{dt}$. The resulting system is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= F(x, y, t).\end{aligned}$$

Autonomous systems

If f and g have no explicit dependence on the independent variable t then the system (7-8) is **autonomous**. In this case the equations for the orbits are given implicitly by the solution of

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.$$