MATH/PHYS 4530 Final Exam - Take Home Problems

Fall 2011

Dr. Robinson Name ______________________

Due 10:00 am, Monday, December 5.

Instructions: You must work alone on this portion of the test. You may use your text or any other books you can find, but you must provide references if you do so. You may use calculators, Mathematica, Maple, or other computer programs, but you should document any such use. You may use your class notes and you may ask questions of me. You *may not* discuss this portion of the test with anyone else. Sign in the space provided above to indicate that you understand and have followed these directions, and return this sheet to me with your solutions.

1. Consider the one-dimensional heat flow equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

with boundary conditions

$$u(0,t) = u(l,t) = 0$$

and initial condition

$$u(x,0) = \frac{100}{l}x.$$

(a) Let $U(x,p)$ denote the Laplace transform of $u(x,t)$ with respect to t. Show that $U(x,p)$ satisfies

$$\frac{\partial^2 U}{\partial x^2} - \frac{p}{\alpha^2}U = -\frac{100}{\alpha^2 l}x$$

and

$$U(0,p) = U(l,p) = 0.$$

(b) Solve the differential equation in part (a) to obtain

$$U(x,p) = -\frac{100\sinh(\sqrt{p}x/\alpha)}{p\sinh(\sqrt{p}l/\alpha)} + \frac{100}{pl}x.$$

(c) Find $u(x,t)$, the inverse Laplace transform of $U(x,p)$ in part (b), by first expanding $U(x,p)$ as a function of x in a Fourier sine series on $(0,l)$ and then looking up the inverse transform of the individual terms in the series.

2. The Klein-Gordon equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + \lambda^2 u$$

arises in quantum mechanics. In two spatial dimensions this equation also describes the vibration of a stretched membrane embedded in an elastic medium. Solve completely the two-dimensional Klein-Gordon equation on a circular membrane of radius a with $u = 0$ on the boundary and $u = f(r,\theta)$ and $\partial u/\partial t = 0$ at time $t = 0$, that is, find the general solution as a series and give formulas for the coefficients of the series in terms of the initial condition.