Math/Phys 4530 Homework #1 Dr. S. Robinson

Fall 2011

Solutions

Some of these problems are adapted from Riley, Hobson, and Bence, *Mathematical Methods for Physics and Engineering,* and Arfken and Weber, *Mathematical Methods for Physicists.*

1. Planck's theory of quantized oscillators leads to an average energy

$$\langle \varepsilon \rangle = \frac{\sum\limits_{n=1}^{\infty} n\varepsilon_0 e^{-n\varepsilon_0/(kT)}}{\sum\limits_{n=0}^{\infty} e^{-n\varepsilon_0/(kT)}}$$

where $\varepsilon_0 = h\nu$, h is the Planck constant, k is the Boltzmann constant, and T is the absolute temperature.

(a) Prove that the series in the denominator converges and

$$\sum_{n=0}^{\infty} e^{-n\varepsilon_0/(kT)} = \frac{1}{1 - e^{-\varepsilon_0/(kT)}}.$$

The series is geometric

$$\sum_{n=0}^{\infty} e^{-n\varepsilon_0/(kT)} = \sum_{n=0}^{\infty} \left(e^{-\varepsilon_0/(kT)}\right)^n$$

and, since $0 < e^{-\varepsilon_0/(kT)} < 1$, it converges with sum given by equation (1.2) on page 2 of the text:

$$\sum_{n=0}^{\infty} e^{-n\varepsilon_0/(kT)} = \frac{1}{1 - e^{-\varepsilon_0/(kT)}}.$$

(b) Prove that the series in the numerator converges and

$$\sum_{n=1}^{\infty} n\varepsilon_0 e^{-n\varepsilon_0/(kT)} = \frac{e^{-\varepsilon_0/(kT)}\varepsilon_0}{(e^{-\varepsilon_0/(kT)} - 1)^2}.$$

Differentiating both sides of the known summation formula (for $|x| < 1$)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

gives

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}.$$

Multiplying both sides by x,

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} = \frac{x}{(x-1)^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} n\varepsilon_0 e^{-n\varepsilon_0/(kT)} = \varepsilon_0 \sum_{n=1}^{\infty} n \left(e^{-\varepsilon_0/(kT)}\right)^n = \varepsilon_0 \frac{e^{-\varepsilon_0/(kT)}}{(e^{-\varepsilon_0/(kT)} - 1)^2}.$$

(c) Conclude from parts (a) and (b) that

$$\langle \varepsilon \rangle = \frac{\varepsilon_0}{e^{\varepsilon_0/(kT)} - 1}.$$

From the above,

$$\langle \varepsilon \rangle = \frac{\varepsilon_0 \frac{e^{-\varepsilon_0/(kT)}}{(e^{-\varepsilon_0/(kT)}-1)^2}}{\frac{1}{1-e^{-\varepsilon_0/(kT)}}} = \frac{\varepsilon_0 e^{-\varepsilon_0/(kT)}}{1 - e^{-\varepsilon_0/(kT)}} = \frac{\varepsilon_0}{e^{\varepsilon_0/(kT)} - 1}.$$

(d) Show that, if $kT >> \varepsilon_0$ then $\langle \varepsilon \rangle \approx kT$, while if $kT << \varepsilon_0$ then $\langle \varepsilon \rangle \approx \varepsilon_0 e^{-\varepsilon_0/(kT)}$.

First, note that if $kT >> \varepsilon_0$ then $\varepsilon_0/(kT) << 1$. Now, by l'Hôpital's rule,

$$\lim_{x \to 0} \frac{x}{e^x - 1} = \lim_{x \to 0} \frac{1}{e^x} = 1$$

so

$$\begin{aligned} \lim_{\varepsilon_0/(kT) \to 0} \langle \varepsilon \rangle &= \lim_{\varepsilon_0/(kT) \to 0} \frac{\varepsilon_0}{e^{\varepsilon_0/(kT)} - 1} \\ &= kT \lim_{\varepsilon_0/(kT) \to 0} \frac{\varepsilon_0/(kT)}{e^{\varepsilon_0/(kT)} - 1} \\ &= kT \end{aligned}$$

and hence, $\langle \varepsilon \rangle \approx kT$ if $kT >> \varepsilon_0$.
If $kT << \varepsilon_0$ then $\varepsilon_0/(kT) >> 1$. Note that

$$\lim_{\varepsilon_0/(kT) \to \infty} e^{\varepsilon_0/(kT)} \langle \varepsilon \rangle = \lim_{\varepsilon_0/(kT) \to \infty} \frac{\varepsilon_0 e^{\varepsilon_0/(kT)}}{e^{\varepsilon_0/(kT)} - 1} = \varepsilon_0$$

so $e^{\varepsilon_0/(kT)} \langle \varepsilon \rangle \approx \varepsilon_0$, that is, $\langle \varepsilon \rangle \approx \varepsilon_0 e^{-\varepsilon_0/(kT)}$, if $kT << \varepsilon_0$.

2. (a) Problem 27 on page 81 of the text.

If $z = x + iy$ with x and y real, then $\bar{z} = x - iy$ and

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(x + iy + x - iy) = x = \operatorname{Re} z$$

$$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(x + iy - (x - iy)) = y = \operatorname{Im} z.$$

Also,

$$|e^z|^2 = e^z \overline{e^z} = e^{x+iy} e^{x-iy} = e^{2x} = e^{2 \operatorname{Re} z}.$$

A little algebra shows that

$$\begin{aligned} (1 + ix)^2(1 - it) - |1 + it|^2 &= (1 + ix)^2(1 - it) - (1 + it)(1 - it) \\ &= -t^2 + 2xt - x^2 + i(tx^2 + 2x - t) \\ &= -(t - x)^2 + i(tx^2 + 2x - t) \end{aligned}$$

so

$$\left| e^{(1+ix)^2(1-it)-|1+it|^2} \right|^2 = e^{-2(t-x)^2}.$$

(b) Prove the identity

$$\left(\frac{ia - 1}{ia + 1} \right)^{ib} = e^{-2b \cot^{-1} a}$$

in which a and b are real. This identity arises in the quantum theory of photoionization.

We first rewrite

$$\begin{aligned}
\frac{ia-1}{ia+1} &= \frac{ia-1}{ia+1} \times \frac{-ia+1}{-ia+1} \\
&= \frac{a^2-1+2ia}{a^2+1} \\
&= \frac{a^2-1}{a^2+1} + i\frac{2a}{a^2+1} \\
&= re^{i\theta}
\end{aligned}$$

where

$$r = \sqrt{\left(\frac{a^2-1}{a^2+1}\right)^2 + \left(\frac{2a}{a^2+1}\right)^2} = 1$$

and θ is such that

$$\sin\theta = \frac{2a}{a^2+1}$$

and

$$\cos\theta = \frac{a^2-1}{a^2+1}.$$

Then,

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = e^{-b\theta},$$

and making use of the trigonometric identity

$$\begin{aligned}
\cot\frac{\theta}{2} &= \frac{\sin\theta}{1-\cos\theta} \\
&= \frac{\frac{2a}{a^2+1}}{1-\frac{a^2-1}{a^2+1}} \\
&= a
\end{aligned}$$

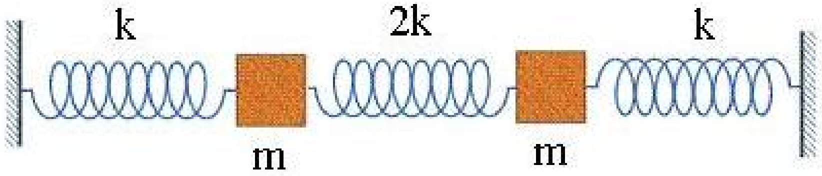
we have

$$\theta = 2\operatorname{arccot} a$$

which gives the desired result.

3. Find the characteristic frequencies and characteristic modes of vibration for the following systems of masses and springs:

   (a)



Let x and y be the coordinates of the two masses at time t relative to their equilibrium positions. The potential energy of the system is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}2k(x-y)^2 + \frac{1}{2}ky^2 = \frac{1}{2}k\left(3x^2 - 4xy + 3y^2\right)$$

and the equations of motion are

$$\begin{aligned} m\ddot{x} &= -\frac{\partial V}{\partial x} = -\frac{1}{2}k\,(6x-4y) = k(-3x+2y) \\ m\ddot{y} &= -\frac{\partial V}{\partial y} = -\frac{1}{2}k\,(-4x+6y) = k(2x-3y). \end{aligned}$$

Assuming solutions of the form $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$ leads to

$$\begin{aligned} -m\omega^2 x_0 &= k(-3x_0+2y_0) \\ -m\omega^2 y_0 &= k(2x_0-3y_0) \end{aligned}$$

or

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \text{ with } \lambda = \frac{m\omega^2}{k}.$$

The eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

are given by the solutions of

$$\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 4 = 0,$$

namely, $\lambda = 1$ or $\lambda = 5$. The characteristic frequencies are therefore

$$\omega = \sqrt{\frac{\lambda k}{m}} = \sqrt{\frac{k}{m}} \text{ or } \sqrt{\frac{5k}{m}}.$$

To find the eigenvectors for we solve the systems

$$\begin{bmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $\lambda = 1$ this is

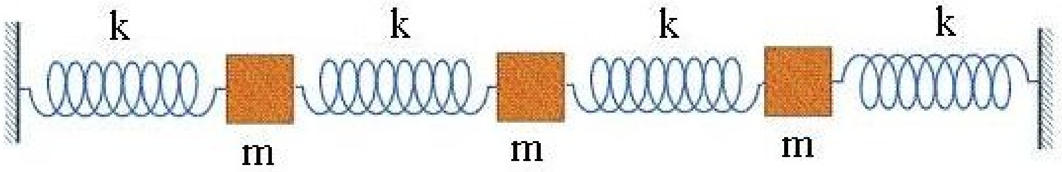
$$\begin{aligned} 2x_0 - 2y_0 &= 0 \\ -2x_0 + 2y_0 &= 0 \end{aligned}$$

which has solution $x_0 = y_0$ corresponding to an eigenvector $[1,1]$ which represents a motion where the masses oscillate back and forth in the same direction.
For $\lambda = 5$ the system is

$$\begin{aligned} -2x_0 - 2y_0 &= 0 \\ -2x_0 - 2y_0 &= 0 \end{aligned}$$

which has solution $x_0 = -y_0$ corresponding to an eigenvector $[1,-1]$ which represents a motion where the masses oscillate back and forth in opposite directions.

(b)



Let x, y, and z be the coordinates of the two masses at time t relative to their equilibrium positions. The potential energy of the system is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k(x-y)^2 + \frac{1}{2}k(y-z)^2 + \frac{1}{2}kz^2 = k\left(x^2 - xy + y^2 - yz + z^2\right)$$

and the equations of motion are

$$\begin{aligned} m\ddot{x} &= -\frac{\partial V}{\partial x} = -k\,(2x - y) = k(-2x + y) \\ m\ddot{y} &= -\frac{\partial V}{\partial y} = -k\,(-x + 2y - z) = k(x - 2y + z) \\ m\ddot{z} &= -\frac{\partial V}{\partial z} = -k\,(-y + 2z) = k(y - 2z) \end{aligned}$$

Assuming solutions of the form $x = x_0 e^{i\omega t}$, $y = y_0 e^{i\omega t}$, $z = z_0 e^{i\omega t}$ leads to

$$\begin{aligned} -m\omega^2 x_0 &= k(-2x_0 + y_0) \\ -m\omega^2 y_0 &= k(x_0 - 2y_0 + z_0) \\ -m\omega^2 z_0 &= k(y_0 - 2z_0) \end{aligned}$$

or

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \text{ with } \lambda = \frac{m\omega^2}{k}.$$

The eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

are given by the roots of

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda)\left((2-\lambda)^2 - 1\right) - (-1)(-1)(2-\lambda) \\ &= (2-\lambda)(\lambda^2 - 4\lambda + 2) \end{aligned}$$

namely, $\lambda = 2$ or $\lambda = 2 \pm \sqrt{2}$. The characteristic frequencies are therefore

$$\omega = \sqrt{\frac{\lambda k}{m}} = \sqrt{\frac{2k}{m}} \text{ or } \sqrt{\frac{(2+\sqrt{2})k}{m}} \text{ or } \sqrt{\frac{(2-\sqrt{2})k}{m}}.$$

To find the eigenvectors for we solve the systems

$$\begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

First, for $\lambda = 2$,

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $y_0 = 0$ and $x_0 = -z_0$, so the eigenvectors in this case are multiples of

$$\overrightarrow{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

which represents a motion with the central mass fixed and the two outer masses oscillating in opposite directions.

For the cases where $\lambda = (2 \pm \sqrt{2})$,

$$\begin{bmatrix} \mp\sqrt{2} & -1 & 0 \\ -1 & \mp\sqrt{2} & -1 \\ 0 & -1 & \mp\sqrt{2} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and it is not too difficult to see that the eigenvectors in this case are multiples of

$$\overrightarrow{v} = \begin{bmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{bmatrix}.$$

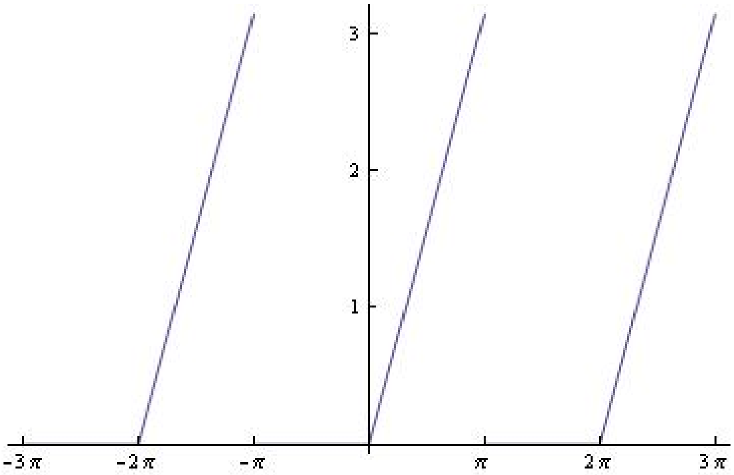
In both cases the outer masses oscillate in the same direction while, depending on the sign of the second component, the inner mass oscillates either in the same or opposite direction.

4. Problem 7 on page 355 of the text. Then expand the function in a series of complex exponentials e^{inx} on $(-\pi, \pi)$ and verify that the two series are equivalent.

   For

   $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

   the graph of the periodic extension looks like:



The Fourier coefficients in the expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

are given by equations (5.9) and (5.10) on page 352 as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\ dx = \frac{(-1)^n - 1}{n^2\pi} \\ a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{\pi}\int_{0}^{\pi} x\ dx = \frac{\pi}{2} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\ dx = -\frac{(-1)^n}{n} \end{aligned}$$

where I used *Mathematica* to evaluate some of the integrals. The Fourier expansion for (the periodic extension of) $f(x)$ is

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi}\cos nx - \frac{(-1)^n}{n}\sin nx\right).$$

The Fourier coefficients in the complex expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

are given by equation (7.6) on page 359 as

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx = \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}\ dx = \frac{(-1)^n - 1}{2n^2\pi} + i\frac{(-1)^n}{n} \\
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{0}^{\pi} x\ dx = \frac{\pi}{4}
\end{aligned}$$

where I used *Mathematica* to evaluate the first integral. The complex Fourier expansion for (the periodic extension of) $f(x)$ is

$$f(x) = \frac{\pi}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{(-1)^n - 1}{2n^2\pi} + i\frac{(-1)^n}{n}\right) e^{inx}.$$

To see that this agrees with the result above, note that

$$\begin{aligned}
f(x) &= \frac{\pi}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{(-1)^n - 1}{2n^2\pi} + i\frac{(-1)^n}{n}\right) e^{inx} \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{2n^2\pi} + i\frac{(-1)^n}{n}\right) e^{inx} + \sum_{n=-\infty}^{-1} \left(\frac{(-1)^n - 1}{2n^2\pi} + i\frac{(-1)^n}{n}\right) e^{inx} \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{2n^2\pi} + i\frac{(-1)^n}{n}\right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{-n} - 1}{2(-n)^2\pi} + i\frac{(-1)^{-n}}{(-n)}\right) e^{-inx} \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{2n^2\pi}(e^{inx} + e^{-inx}) + i\frac{(-1)^n}{n}(e^{inx} - e^{-inx})\right) \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{n^2\pi}\cos nx - \frac{(-1)^n}{n}\sin nx\right).
\end{aligned}$$

5. Problem 23 on page 371 of the text.

The initial position of the string can be expressed as

$$f(x,0) = F(x) = \begin{cases} \frac{2h}{l}x, & 0 \le x \le \frac{l}{2} \\ \frac{2h}{l}(l-x), & \frac{l}{2} \le x \le l \end{cases}.$$

The coefficients b_n in the Fourier sine series

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

are given by formula (9.4) in the text:

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left(\int_0^{l/2} \frac{2h}{l}x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} \frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}dx\right) \\
&= \frac{4h}{l^2}\left(\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}dx\right) \\
&= \frac{4h}{l^2}\left(\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\frac{n\pi}{2}\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} - \sin n\pi\right)\right) \\
&= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}$$

where I used *Mathematica* to do the integrations though they are easily done by parts. So,

$$\begin{aligned}
F(x) &= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l} \\
&= \frac{8h}{\pi^2}\left(\sin\frac{\pi x}{l}-\frac{1}{9}\sin\frac{3\pi x}{l}+\frac{1}{25}\sin\frac{5\pi x}{l}-\cdots\right) \\
&= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}.
\end{aligned}$$