

Solutions

1. Problem 24 on page 385 of the text.

The exponential Fourier transform of $f(x) = e^{-|x|}$ is

$$\begin{aligned}
 g(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\alpha x} dx \\
 &= \frac{1}{2\pi} \left(\int_{-\infty}^0 e^x e^{-i\alpha x} dx + \int_0^{\infty} e^{-x} e^{-i\alpha x} dx \right) \\
 &= \frac{1}{2\pi} \left(\int_{-\infty}^0 e^{(1-i\alpha)x} dx + \int_0^{\infty} e^{-(1+i\alpha)x} dx \right) \\
 &= \frac{1}{2\pi} \left(\frac{1}{(1-i\alpha)} e^{(1-i\alpha)x} \Big|_{-\infty}^0 - \frac{1}{(1+i\alpha)} e^{-(1+i\alpha)x} \Big|_0^{\infty} \right) \\
 &= \frac{1}{2\pi} \left(\frac{1}{(1-i\alpha)} + \frac{1}{(1+i\alpha)} \right) \\
 &= \frac{1}{\pi(1+\alpha^2)}.
 \end{aligned}$$

From the inverse transform we obtain

$$e^{-|x|} = \int_{-\infty}^{\infty} \frac{1}{\pi(1+\alpha^2)} e^{i\alpha x} d\alpha$$

so

$$\begin{aligned}
 \pi e^{-|x|} &= \int_{-\infty}^{\infty} \frac{1}{1+\alpha^2} e^{i\alpha x} d\alpha \\
 &= \int_{-\infty}^{\infty} \frac{1}{1+\alpha^2} \cos \alpha x d\alpha + i \int_{-\infty}^{\infty} \frac{1}{1+\alpha^2} \sin \alpha x d\alpha \\
 &= 2 \int_0^{\infty} \frac{1}{1+\alpha^2} \cos \alpha x d\alpha + 0
 \end{aligned}$$

since the cosine and sine are even and odd functions, respectively. This gives the desired result

$$\int_0^{\infty} \frac{\cos \alpha x}{1+\alpha^2} d\alpha = \frac{\pi}{2} e^{-|x|}. \quad (1)$$

The Fourier cosine transform of $f(x) = e^{-|x|}$ is

$$\begin{aligned}
 g_c(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos \alpha x dx \\
 &= \sqrt{\frac{2}{\pi}} \left(\frac{e^{-x}}{1+\alpha^2} (\alpha \sin \alpha x - \cos \alpha x) \Big|_0^{\infty} \right) \\
 &= \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}
 \end{aligned}$$

and the inverse transform gives

$$\begin{aligned}
 e^{-|x|} &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2} \cos \alpha x d\alpha \\
 &= \frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{1+\alpha^2} d\alpha
 \end{aligned}$$

in agreement with equation (1).

The Fourier cosine transform of $f(x) = 1/(1+x^2)$ is

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos \alpha x \, dx.$$

From equation (1) with x and α interchanged we have

$$\int_0^\infty \frac{1}{1+x^2} \cos \alpha x \, dx = \frac{\pi}{2} e^{-|\alpha|}$$

so

$$g_c(\alpha) = \sqrt{\frac{\pi}{2}} e^{-|\alpha|}.$$

2. Problem 18 on page 449 of the text.

Taking the Laplace transform of both sides of the differential equation

$$y'' + \omega^2 y = f(t)$$

and using the initial conditions gives

$$p^2 Y(p) + \omega^2 Y(p) = F(p) \tag{2}$$

where Y and F denote the Laplace transforms of y and

$$f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & \text{otherwise} \end{cases},$$

respectively. Solving (2) for $Y(p)$

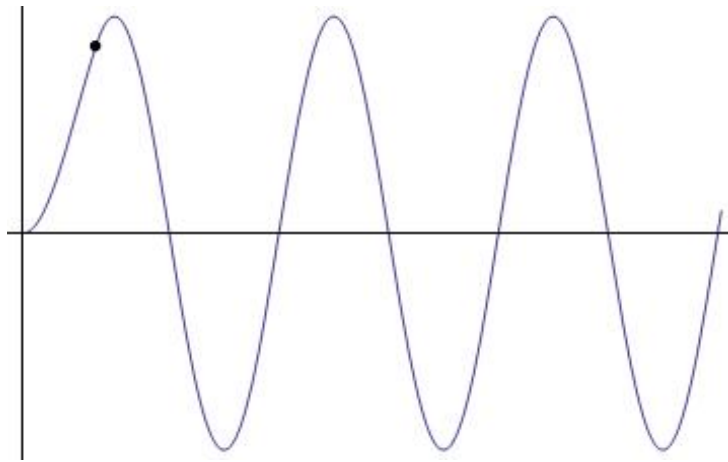
$$Y(p) = \frac{1}{p^2 + \omega^2} F(p)$$

and using *L34* and *L3* on the table of Laplace transforms gives

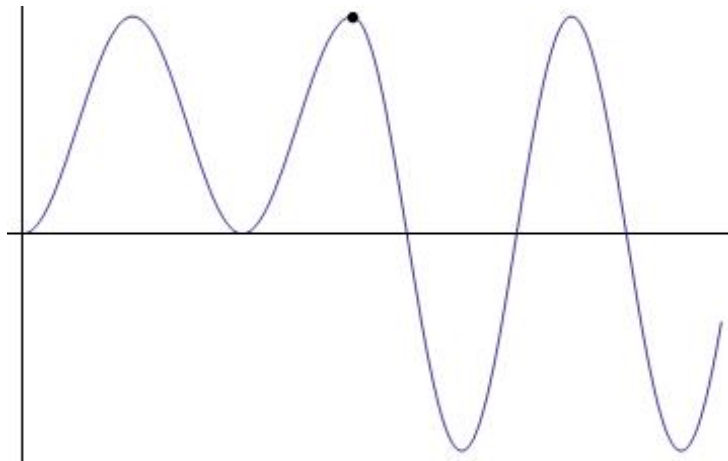
$$\begin{aligned} y(t) &= \frac{1}{\omega} \int_0^t \sin(\omega(t-\tau)) f(\tau) \, d\tau \\ &= \begin{cases} \frac{1}{\omega} \int_0^t \sin(\omega(t-\tau)) \, d\tau, & 0 < t < a \\ \frac{1}{\omega} \int_0^a \sin(\omega(t-\tau)) \, d\tau, & t > a \end{cases} \\ &= \begin{cases} \frac{1 - \cos \omega t}{\omega^2}, & 0 < t < a \\ \frac{\cos(\omega(t-a)) - \cos \omega t}{\omega^2}, & t > a \end{cases}. \end{aligned}$$

Sketches of the motion for $a = \frac{1}{3}T = \frac{2\pi}{3\omega}$, $a = \frac{3}{2}T = \frac{3\pi}{\omega}$, and $a = \frac{1}{10}T = \frac{\pi}{5\omega}$ follow. In each case, the point $(a, y(a))$ is plotted for reference.

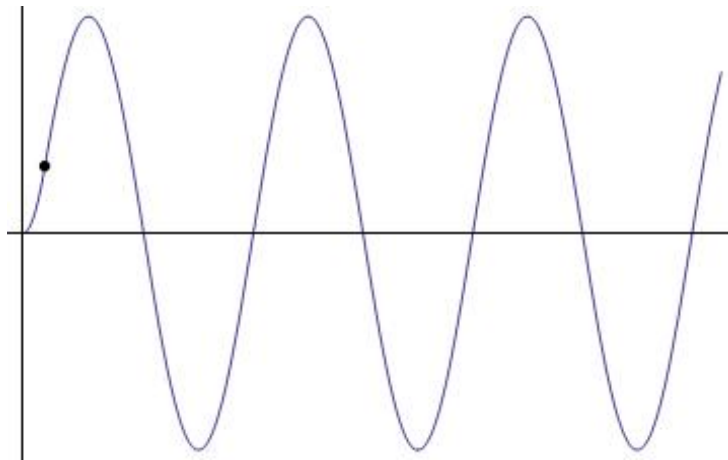
$$a = \frac{1}{3}T :$$



$$a = \frac{3}{2}T :$$



$$a = \frac{1}{10}T :$$



3. Problem 9 on page 459 of the text.

Taking the Laplace transform of both sides of the differential equation

$$y'' + 2y' + 10y = \delta(t - t_0)$$

and using the initial conditions $y(0) = y'(0) = 0$ gives

$$p^2Y(p) + 2pY(p) + 10Y(p) = e^{-pt_0}$$

where Y denotes the Laplace transform of y . Solving for Y ,

$$Y(p) = \frac{1}{p^2 + 2p + 10} e^{-pt_0}.$$

Note that

$$p^2 + 2p + 10 = (p + 1)^2 + 9$$

so, using *L13* on the table of Laplace transforms, the inverse Laplace transform of the transfer function $1/(p^2 + 2p + 10)$ is

$$g(t) = \frac{1}{3} e^{-t} \sin 3t.$$

Then, using *L28* on the table of Laplace transforms,

$$y(t) = \begin{cases} 0, & 0 < t < t_0 \\ \frac{1}{3} e^{-(t-t_0)} \sin 3(t-t_0), & t > t_0 \end{cases}.$$

4. Problem 4 on page 567 of the text.

Given that $P_l(x)$ is a solution of the differential equation

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0 \tag{3}$$

we can find a second linearly independent solution using the method of reduction of order, that is, by seeking a solution of the form

$$Q_l(x) = v(x)P_l(x)$$

where $v(x)$ is a function to be determined. Substituting $y = Q_l = vP_l$,

$$y' = Q_l' = v'P_l + vP_l',$$

and

$$y'' = Q_l'' = v''P_l + 2v'P_l' + vP_l'',$$

into (3),

$$\begin{aligned} (1 - x^2)(v''P_l + 2v'P_l' + vP_l'') - 2x(v'P_l + vP_l') + l(l + 1)vP_l &= 0 \\ (1 - x^2)(v''P_l + 2v'P_l') - 2x(v'P_l) + v[(1 - x^2)P_l'' - 2xP_l' + l(l + 1)P_l] &= 0 \\ (1 - x^2)P_lv'' + [2(1 - x^2)P_l' - 2xP_l]v' &= 0. \end{aligned}$$

This last equation is a first-order linear differential equation for $w(x) = v'(x)$:

$$\begin{aligned} (1 - x^2)P_lw' + [2(1 - x^2)P_l' - 2xP_l]w &= 0 \\ w' + \left(\frac{2(1 - x^2)P_l' - 2xP_l}{(1 - x^2)P_l} \right)w &= 0 \end{aligned}$$

having a solution

$$w(x) = \exp\left(-\int \frac{2(1 - x^2)P_l'(x) - 2xP_l(x)}{(1 - x^2)P_l(x)} dx\right)$$

and therefore

$$v(x) = \int w(x) dx = \int \exp\left(-\int \frac{2(1 - x^2)P_l'(x) - 2xP_l(x)}{(1 - x^2)P_l(x)} dx\right) dx$$

and

$$Q_l(x) = v(x)P_l(x) = P_l(x) \int \exp\left(-\int \frac{2(1 - x^2)P_l'(x) - 2xP_l(x)}{(1 - x^2)P_l(x)} dx\right) dx.$$

In the case $l = 0$ this gives

$$\begin{aligned}
 Q_0(x) &= \int \exp\left(\int \frac{2x}{(1-x^2)} dx\right) dx \\
 &= \int e^{-\log|x^2-1|} dx \\
 &= \int \frac{1}{1-x^2} dx \\
 &= \frac{1}{2} \log \frac{1+x}{1-x}.
 \end{aligned}$$

Note that the MacLauren series for this last expression is

$$\frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

which agrees with the a_1 series with $l = 0$ in equation (2.7) of the text. Also note that this series diverges at $x = 1$.

In the case $l = 1$ we have (using Mathematica to evaluate the integrals)

$$\begin{aligned}
 Q_1(x) &= x \int \exp\left(-\int \frac{4x^2-2}{x^3-x} dx\right) dx \\
 &= x \int e^{-\log x^2(1-x^2)} dx \\
 &= x \int \frac{1}{x^2(1-x^2)} dx \\
 &= \frac{x}{2} \log \frac{1+x}{1-x} - 1.
 \end{aligned}$$

5. Problem 23 on page 617 of the text.

Substituting

$$\begin{aligned}
 y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 y'(x) &= \sum_{n=0}^{\infty} a_n n x^{n-1} \\
 y''(x) &= \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}
 \end{aligned}$$

into the Chebyshev equation

$$(1-x^2)y'' - xy' + k^2y = 0$$

leads to

$$\begin{aligned}
 (1-x^2) \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} - x \sum_{n=0}^{\infty} a_n n x^{n-1} + k^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
 \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1) x^n - \sum_{n=0}^{\infty} a_n n x^n + k^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
 \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n - \sum_{n=0}^{\infty} a_n n(n-1) x^n - \sum_{n=0}^{\infty} a_n n x^n + k^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
 \sum_{n=0}^{\infty} (a_{n+2}(n+2)(n+1) - a_n(n^2 - k^2)) x^n &= 0
 \end{aligned}$$

so

$$a_{n+2}(n+2)(n+1) - a_n(n^2 - k^2) = 0$$

for all n , or

$$a_{n+2} = a_n \frac{(n^2 - k^2)}{(n+2)(n+1)}.$$

The first few coefficients are

$$\begin{aligned} a_2 &= -\frac{1}{2}k^2 a_0 \\ a_3 &= \frac{1}{6}(1 - k^2)a_1 \\ a_4 &= -\frac{1}{24}k^2(4 - k^2)a_0 \\ a_5 &= \frac{1}{120}(1 - k^2)(9 - k^2)a_1 \\ a_6 &= -\frac{1}{6!}k^2(2^2 - k^2)(4^2 - k^2)a_0 \\ a_7 &= \frac{1}{7!}(1^2 - k^2)(3^2 - k^2)(5^2 - k^2)a_1 \end{aligned}$$

from which a pattern is evident. Our solution is therefore

$$\begin{aligned} y(x) &= a_0 \left(1 - \frac{1}{2}k^2 x^2 - \frac{1}{24}k^2(4 - k^2)x^4 - \frac{1}{6!}k^2(2^2 - k^2)(4^2 - k^2)a_0 + \dots \right) + \\ &\quad a_1 \left(x + \frac{1}{6}(1 - k^2)x^3 + \frac{1}{120}(1 - k^2)(9 - k^2)x^5 + \frac{1}{7!}(1^2 - k^2)(3^2 - k^2)(5^2 - k^2)x^7 + \dots \right). \end{aligned}$$

Note that if k is an integer then one of the two series above is finite and therefore a polynomial. For $k = 0, 1,$ and $2,$ these polynomials are $a_0, a_1 x,$ and $a_0(1 - 2x^2),$ respectively. Choosing a_0 or a_1 so that the result evaluates to 1 when $x = 1$ gives the Chebyshev polynomials

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1. \end{aligned}$$