Math/Phys 4530 Homework #3 Dr. S. Robinson

Fall 2011

Solutions

1. Problem #12 on page 627 of the text.

   We choose our coordinate system so that the plate lies in the region $0 \le x \le 10$, $0 \le y \le 30$ and the temperature $u(x, y)$ of the plate satisfies

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   with the boundary conditions

$$\begin{aligned} u(x,0) &= 0 \\ u(x,30) &= 100 \\ u(0,y) &= 0 \\ u(10,y) &= 100. \end{aligned}$$

   We decompose the problem into two simpler problems by letting $u(x, y) = v(x, y) + w(x, y)$ where v and w satisfy

$$\nabla^2 v = \nabla^2 w = 0$$

   with boundary conditions

$$\begin{aligned} v(x,0) &= 0 \\ v(x,30) &= 100 \\ v(0,y) &= 0 \\ v(10,y) &= 0 \end{aligned}$$

   and

$$\begin{aligned} w(x,0) &= 0 \\ w(x,30) &= 0 \\ w(0,y) &= 0 \\ w(10,y) &= 100. \end{aligned}$$

   Following page 622 of the text, v is a linear combination of separated solutions having the form $X(x)Y(y)$ where

$$X(x) = \left\{ \begin{array}{l} \sin kx \\ \cos kx \end{array} \right. \quad \text{and } Y(y) = \left\{ \begin{array}{l} \sinh ky \\ \cosh ky \end{array} \right.$$

   for some constant $k > 0$. The boundary condition $X(0) = X(10) = 0$ gives $k = n\pi/10$ and $X(x) = \sin \frac{n\pi x}{10}$. The boundary condition $Y(0) = 0$ then gives $Y(y) = \sinh \frac{n\pi y}{10}$, so

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{10} \sinh \frac{n\pi y}{10}$$

   and the remaining boundary condition $v(x, 30) = 100$ gives

$$100 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{10} \sinh 3n\pi$$

   hence, from equation (9.4) on page 366 (the coefficients of a Fourier sine series)

$$a_n \sinh 3n\pi = \frac{1}{5} \int_0^{10} 100 \sin \frac{n\pi x}{10}\, dx = \frac{200}{n\pi} \left(1 - (-1)^n\right)$$

and

$$v(x,y)=\sum_{n=1}^{\infty}\frac{200\,(1-(-1)^n)}{n\pi\sinh 3n\pi}\sin\frac{n\pi x}{10}\sinh\frac{n\pi y}{10}.$$

Separating the problem for w leads to separated solutions having the form $X(x)Y(y)$ where

$$X(x)=\begin{cases}\sinh kx\\ \cosh kx\end{cases}\quad\text{and } Y(y)=\begin{cases}\sin ky\\ \cos ky\end{cases}$$

for some constant $k>0$. The boundary condition $Y(0)=Y(30)=0$ gives $k=n\pi/30$ and $Y(y)=\sin\frac{n\pi y}{30}$. The boundary condition $X(0)=0$ then gives $X(x)=\sinh\frac{n\pi x}{30}$, so

$$w(x,y)=\sum_{n=1}^{\infty}b_n\sinh\frac{n\pi x}{30}\sin\frac{n\pi y}{30}$$

and the remaining boundary condition $w(10,y)=100$ gives

$$100=\sum_{n=1}^{\infty}b_n\sinh\frac{n\pi}{3}\sin\frac{n\pi y}{30}$$

and again from equation (9.4) on page 366

$$b_n\sinh\frac{n\pi}{3}=\frac{1}{15}\int_0^{30}100\sin\frac{n\pi x}{30}\,dx=\frac{200}{n\pi}\left(1-(-1)^n\right)$$

and

$$w(x,y)=\sum_{n=1}^{\infty}\frac{200\,(1-(-1)^n)}{n\pi\sinh\frac{n\pi}{3}}\sinh\frac{n\pi x}{30}\sin\frac{n\pi y}{30}.$$

Finally, the solution to our original problem is

$$\begin{aligned}u(x,y)&=v(x,y)+w(x,y)\\&=200\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n\pi}\left(\frac{1}{\sinh 3n\pi}\sin\frac{n\pi x}{10}\sinh\frac{n\pi y}{10}+\frac{1}{\sinh\frac{n\pi}{3}}\sinh\frac{n\pi x}{30}\sin\frac{n\pi y}{30}\right).\end{aligned}$$

2. Problem #5 on page 637 of the text.

   The displacement $y(x,t)$ of the string satisfies

$$\begin{aligned}\frac{\partial^2y}{\partial x^2}&=\frac{1}{v^2}\frac{\partial^2y}{\partial t^2}\\y(0,t)&=0\\y(l,t)&=0\\y(x,0)&=0\\\frac{\partial y}{\partial t}(x,0)&=V(x).\end{aligned}$$

   From equations (4.10) in Chapter 13 we know that

$$y(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}$$

   so

$$\frac{\partial y}{\partial t}(x,0)=V(x)=\sum_{n=1}^{\infty}B_n\frac{n\pi v}{l}\sin\frac{n\pi x}{l}.$$

Using the result of Problem #5 from the first homework assignment

$$B_n \frac{n\pi v}{l} = \frac{2}{l}\int_0^l V(x)\sin\frac{n\pi x}{l}\,dx = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

and therefore

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8hl}{n^3\pi^3 v}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi v t}{l}.$$

3. Problem #3 on page 643 of the text.

We choose coordinates so that the temperature $u(r,\theta,z)$ satisfies

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

with boundary conditions

$$\begin{aligned} u(a,\theta,z) &= 0 \\ u(r,\theta,H) &= 0 \\ u(r,\theta,0) &= 100. \end{aligned}$$

Seeking separated solutions of the form $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ and noticing that the solution must be independent of θ gives, by following Section 5 in Chapter 13,

$$R(r) = J_0(kr/a) \text{ and } Z(z) = \begin{cases} e^{kz/a} \\ e^{-kz/a} \end{cases}.$$

Anticipating using the boundary condition at $z = H$ we rewrite the Z solution as

$$Z(z) = \begin{cases} \cosh(\frac{k}{a}(H-z)) \\ \sinh(\frac{k}{a}(H-z)) \end{cases}$$

and note that $Z(H) = 0$ implies that the hyperbolic cosine term must vanish. From the boundary at $r = a$ we see

$$R(a) = J_0(k) = 0$$

so $k = k_m$ where k_m denotes the m^{th} (positive) zero of $J_0(x)$. Hence,

$$u(r,\theta,z) = \sum_{m=1}^{\infty} a_m J_0(k_m r/a)\sinh(\frac{k_m}{a}(H-z)).$$

From the orthogonality of the Bessel functions, namely, from equation (19.10) in Chapter 12,

$$\int_0^a J_0(k_m r/a)J_0(k_n r/a) r\,dr = a^2\int_0^1 J_0(k_m r)J_0(k_n r) r\,dr = \begin{cases} 0, & m \neq n \\ \frac{a^2}{2}J_1^2(k_n), & m = n \end{cases}$$

and the remaining boundary condition

$$u(r,\theta,0) = 100 = \sum_{m=1}^{\infty} a_m J_0(k_m r/a)\sinh(\frac{k_m}{a}H)$$

we obtain

$$\begin{aligned} 100\int_0^a J_0(k_n r/a) r\,dr &= \sum_{m=1}^{\infty} a_m \sinh(\frac{k_m}{a}H)\int_0^a J_0(k_m r/a)J_0(k_n r/a) r\,dr \\ &= a_n \sinh(\frac{k_n}{a}H)\frac{a^2}{2}J_1^2(k_n). \end{aligned}$$

We can evaluate the integral on the left using equation (5.15) in Chapter 13, resulting in

$$100\frac{a^2}{k_n}J_1(k_n) = a_n \sinh(\frac{k_n}{a}H)\frac{a^2}{2}J_1^2(k_n)$$

or

$$a_n = \frac{200}{k_n \sinh(\frac{k_n}{a}H)J_1(k_n)}$$

and therefore

$$u(r,\theta,z) = \sum_{m=1}^{\infty} \frac{200}{k_m \sinh(\frac{k_m}{a}H)J_1(k_m)} J_0(k_m r/a)\sinh(\frac{k_m}{a}(H-z)).$$

4. Problem #9 on page 643 of the text.

   The temperature $u(x,y,z)$ inside and on the boundary of the cube satisfies

$$\begin{aligned}\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \\ u(0,y,z) = u(10,y,z) = 0 \\ u(x,0,z) = u(x,10,z) = 0 \\ u(x,y,10) = 0 \\ u(x,y,0) = 100.\end{aligned}$$

   Seeking solutions of the form $u(x,y,z) = X(x)Y(y)Z(z)$ with $X(0) = X(10) = Y(0) = Y(10) = Z(10) = 0$ leads to

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

   or

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = \alpha = \text{ constant.}$$

   We know that the nontrivial solutions of $X'' = \alpha X$, $X(0) = X(10) = 0$ are $X(x) = \sin\frac{n\pi x}{10}$ for $\alpha = -\frac{n^2\pi^2}{100}$. Then

$$-\frac{Y''}{Y} = \frac{Z''}{Z} + \alpha = \beta = \text{ constant}$$

   and we know that the nontrivial solutions of $Y'' = -\beta Y$, $Y(0) = Y(10) = 0$ are $Y(y) = \sin\frac{m\pi y}{10}$ for $\beta = \frac{m^2\pi^2}{100}$.So,

$$\begin{aligned}\frac{Z''}{Z} &= \beta - \alpha \\ Z'' &= \frac{\pi^2}{100}(n^2+m^2)Z\end{aligned}$$

   and the solution that satisfies $Z(10) = 0$ is $Z(z) = \sinh(\frac{\pi}{10}\sqrt{n^2+m^2}(10-z))$. Hence our general solution is

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin\frac{n\pi x}{10}\sin\frac{m\pi y}{10}\sinh(\frac{\pi}{10}\sqrt{n^2+m^2}(10-z)).$$

   From the remaining boundary condition,

$$u(x,y,0) = 100 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\frac{n\pi x}{10}\sin\frac{m\pi y}{10}\sinh(\pi\sqrt{n^2+m^2})$$

   and the orthogonality relation

$$\int_0^{10}\int_0^{10} \sin\frac{n\pi x}{10}\sin\frac{p\pi x}{10}\sin\frac{m\pi y}{10}\sin\frac{q\pi y}{10}\,dx\,dy = \begin{cases} 0, & n \neq p \text{ or } m \neq q \\ 25, & n = p \text{ and } m = q\end{cases}$$

we obtain

$$100\int_0^{10}\int_0^{10}\sin\frac{n\pi x}{10}\sin\frac{m\pi y}{10}\;dx\;dy = 25b_{nm}\sinh(\pi\sqrt{n^2+m^2})$$

$$40000\frac{\sin^2(\frac{n\pi}{2})\sin^2(\frac{m\pi}{2})}{mn\pi^2} = 25b_{nm}\sinh(\pi\sqrt{n^2+m^2})$$

so

$$u(x,y,z) = \frac{1600}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin^2(\frac{n\pi}{2})\sin^2(\frac{m\pi}{2})}{mn\sinh(\pi\sqrt{n^2+m^2})}\sin\frac{n\pi x}{10}\sin\frac{m\pi y}{10}\sinh(\frac{\pi}{10}\sqrt{n^2+m^2}(10-z)).$$

5. Problem #15 on page 651 of the text.

If $u(r,\theta,\phi,t)$ is the temperature of the sphere (with radius a), then $v(r,\theta,\phi,t)$ defined by

$$v(r,\theta,\phi,t) = u(r,\theta,\phi,t) - 100$$

satisfies

$$\frac{1}{\alpha^2}\frac{\partial v}{\partial t} = \nabla^2 v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}$$

with the boundary and initial conditions

$$v(a,\theta,\phi,t) = 0,$$
$$v(r,\theta,\phi,0) = -100.$$

From the symmetry of the problem the solution cannot depend on the angle variables. Substituting a solution of the form

$$v(r,\theta,\phi,t) = R(r)T(t)$$

gives

$$\frac{1}{\alpha^2}R\frac{dT}{dt} = T\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

or

$$\frac{1}{\alpha^2 T}\frac{dT}{dt} = \frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right).$$

So, for some constant λ,

$$\frac{1}{\alpha^2 T}\frac{dT}{dt} = \lambda$$

and

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda.$$

From this last equation we have

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda r^2 R.$$

Multiplying both sides by R and integrating by parts,

$$\int_0^a R(r)\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right)dr = \lambda\int_0^a r^2 R(r)^2 dr$$

$$R(r)r^2\frac{d}{dr}R(r)\Big|_0^a - \int_0^a r^2(\frac{d}{dr}R(r))^2 dr = \lambda\int_0^a r^2 R(r)^2 dr.$$

The boundary terms are zero since $R(a) = 0$ and $R'(r)$ is finite, so we have

$$\lambda = -\frac{\int_0^a r^2(\frac{d}{dr}R(r))^2 dr}{\int_0^a r^2 R(r)^2 dr}$$

that is, $\lambda < 0$ if R is not identically zero. So, $\lambda = -K^2$ for some $K > 0$ and, up to a constant factor,

$$T(t) = e^{-K^2\alpha^2 t}.$$

The equation for R is now

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + K^2r^2R = 0$$

which, by equations (16.1)-(16.2) in Chapter 12 of the text has solutions

$$R(r) = \begin{cases} r^{-1/2}J_{1/2}(Kr) \\ r^{-1/2}N_{1/2}(Kr) \end{cases}.$$

Since $N_{1/2}(Kr)$ is badly behaved at $r = 0$, we take only the upper solution, use the boundary condition

$$R(a) = a^{-1/2}J_{1/2}(Ka) = 0$$

to obtain that Ka must be a zero of $J_{1/2}(x)$ and conclude that, up to a constant factor,

$$R(r) = r^{-1/2}J_{1/2}(k_q r/a)$$

where k_q denotes the q^{th} positive zero of $J_{1/2}(x)$. Our solution then must have the form

$$v(r,\theta,\phi,t) = r^{-1/2}\sum_{q=1}^{\infty} c_q J_{1/2}(k_q r/a)e^{-k_q^2\alpha^2 t/a^2}$$

for some choice of constants c_q. Recalling that (see equation (17.4) in Chapter 12)

$$j_0(x) = \sqrt{\frac{\pi}{2x}}J_{1/2}(x) = \frac{\sin x}{x},$$

we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

from which we see

$$k_q = q\pi$$

and

$$v(r,\theta,\phi,t) = \frac{1}{\pi r}\sum_{q=1}^{\infty} c_q\sqrt{\frac{2a}{q}}e^{-q^2\pi^2\alpha^2 t/a^2}\sin\frac{q\pi r}{a}.$$

From the initial condition,

$$\frac{1}{\pi r}\sum_{q=1}^{\infty} c_q\sqrt{\frac{2a}{q}}\sin\frac{q\pi r}{a} = -100$$

so

$$\sum_{q=1}^{\infty} c_q\sqrt{\frac{2a}{q}}\sin\frac{q\pi r}{a} = -100\pi r.$$

Now, the Fourier sine series on $[0, a]$ for $f(r) = -100\pi r$ is

$$-100\pi r = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi r}{a}$$

where

$$b_n = \frac{2}{a}\int_0^a (-100\pi r)\sin\frac{n\pi r}{a}\,dr = \frac{200a}{n}(-1)^n$$

so

$$c_q\sqrt{\frac{2a}{q}} = \frac{200a}{q}(-1)^q$$

and therefore

$$v(r,\theta,\phi,t) = \frac{200a}{\pi r}\sum_{q=1}^{\infty}\frac{(-1)^q}{q}e^{-q^2\pi^2\alpha^2 t/a^2}\sin\frac{q\pi r}{a}$$

or, returning to the original specification of the problem,

$$u(r,\theta,\phi,t) = 100 + \frac{200a}{\pi r}\sum_{q=1}^{\infty}\frac{(-1)^q}{q}e^{-q^2\pi^2\alpha^2 t/a^2}\sin\frac{q\pi r}{a}.$$