Math/Phys 4530 Homework #4 Dr. S. Robinson

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Solutions

1. The mathematical model for the velocity potential $\phi(x,y)$ in the steady, two-dimensional, irrotational flow of an ideal fluid near a square corner is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

for $0 < x < \pi$, $0 < y < \pi$, with the boundary conditions

$$\begin{aligned} \frac{\partial\phi}{\partial x}(0,y) &= 0 \\ \phi(\pi,y) &= c_1 \\ \frac{\partial\phi}{\partial y}(x,0) &= 0 \\ \phi(x,\pi) &= c_2 \end{aligned}$$

where c_1 and c_2 are constants with $c_2 > c_1$. Find $\phi(x,y)$.

First we define $u(x,y) = \phi(x,y) - c_1$ in order to create a homogeneous set of boundary conditions. The function $u(x,y)$ then satisfies

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial x}(0,y) &= 0 \\ u(\pi,y) &= 0 \\ \frac{\partial u}{\partial y}(x,0) &= 0 \\ u(x,\pi) &= c_2 - c_1. \end{aligned}$$

Looking for solutions of the form $u(x,y) = X(x)Y(y)$ leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = \text{ constant}$$

with the boundary conditions $X'(0) = 0$, $X(\pi) = 0$, $Y'(0) = 0$. It follows that the constant λ must be negative if X is not identically zero, for multiplying both sides of the equation for X by X and integrating by parts gives

$$\begin{aligned} \int_0^\pi X(x)X''(x)\,dx &= \lambda\int_0^\pi X(x)^2dx \\ X(\pi)X'(\pi) - X(0)X'(0) - \int_0^\pi X'(x)^2dx &= \lambda\int_0^\pi X(x)^2dx \\ -\int_0^\pi X'(x)^2dx &= \lambda\int_0^\pi X(x)^2dx. \end{aligned}$$

Note that if $X'(x) = 0$ for $0 \le x \le \pi$ then the boundary condition $X(\pi) = 0$ would imply $X(x) = 0$ for $0 \le x \le \pi$, hence, for nontrivial solutions we have

$$\int_0^\pi X'(x)^2dx > 0$$

and

$$\lambda = -\frac{\int_0^\pi X'(x)^2 dx}{\int_0^\pi X(x)^2 dx} < 0.$$

So, letting $\lambda = -K^2$ for some $K > 0$, we have

$$X(x) = \left\{ \begin{array}{l} \sin Kx \\ \cos Kx \end{array} \right. .$$

Since $X'(0) = 0$ we neglect the sine term. Since $X(\pi) = \cos K\pi = 0$ we have $K = \frac{2n-1}{2}$, $n = 1, 2, 3, \ldots$ .Also,

$$Y(y) = \left\{ \begin{array}{l} \sinh Ky \\ \cosh Ky \end{array} \right.$$

but we can negect the hyperbolic sine term since $Y'(0) = 0$. We then have the solution

$$u(x,y) = \sum_{n=1}^\infty a_n \cos\frac{(2n-1)x}{2} \cosh\frac{(2n-1)y}{2}$$

and we need to choose the constants a_n to meet the remaining boundary condition

$$u(x,\pi) = c_2 - c_1 = \sum_{n=1}^\infty a_n \cos\frac{(2n-1)x}{2} \cosh\frac{(2n-1)\pi}{2}.$$

It is not hard to show that

$$\int_0^\pi \cos\frac{(2n-1)x}{2} \cos\frac{(2m-1)x}{2} dx = \left\{ \begin{array}{ll} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{array} \right.$$

so

$$(c_2 - c_1) \int_0^\pi \cos\frac{(2n-1)x}{2} dx = \frac{\pi}{2} a_n \cosh\frac{(2n-1)\pi}{2}$$

or

$$a_n = \frac{4\,(c_2 - c_1)\,(-1)^{n+1}}{\pi(2n-1)\cosh\frac{(2n-1)\pi}{2}}$$

and therefore

$$\phi(x,y) = c_1 + \frac{4}{\pi} \sum_{n=1}^\infty \frac{(c_2 - c_1)\,(-1)^{n+1}}{(2n-1)\cosh\frac{(2n-1)\pi}{2}} \cos\frac{(2n-1)x}{2} \cosh\frac{(2n-1)y}{2}.$$

2. Solve the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi = -\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right)$$

for $\Psi = \Psi(r,\theta,t)$ where $0 < r < a$ and

$$\begin{aligned} \Psi(a,\theta,t) &= 0 \\ \lim_{r\to 0} |\Psi(r,\theta,t)| &< \infty \\ \Psi(r,\theta,0) &= \frac{2}{a^2\sqrt{\pi}} r\sin\theta. \end{aligned}$$

First separating variables as $\Psi(r,\theta,t) = \psi(r,\theta)T(t)$ gives

$$i\hbar\frac{T'(t)}{T(t)} = -\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta) = E$$

where E is a constant and in fact $E > 0$ since the potential energy here is $V(r, \theta) = 0$. Note that then, at least up to a constant factor,

$$T(t) = e^{-iEt/\hbar}.$$

Further separating the space part of the equation with $\psi(r, \theta) = R(r)\Theta(\theta)$ gives

$$-\frac{\hbar^2}{2m}\left(\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta r^2}\frac{\partial^2\Theta}{\partial\theta^2}\right) = E$$

or

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{2m}{\hbar^2}Er^2 = -\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = \text{ constant.}$$

By the periodicity of Θ we know that the constant must have value n^2 for some non-negative integer n and

$$\Theta(\theta) = \left\{ \begin{array}{c} \sin n\theta \\ \cos n\theta \end{array} \right. .$$

The equation for R is then

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\frac{2m}{\hbar^2}Er^2 - n^2\right)R = 0$$

which has as its only bounded solution $R(r) = J_n(r\sqrt{2mE}/\hbar)$. From the boundary condition we then have that

$$a\frac{\sqrt{2mE}}{\hbar} = \kappa_{mn}$$

where κ_{mn} denotes the m^{th} positive zero of $J_n(x)$. So,

$$E = \frac{\hbar^2\kappa_{mn}^2}{2ma^2}$$

the most general solution we have is of the form

$$\Psi_{mn}(r, \theta, t) = e^{-i\hbar\kappa_{mn}^2 t/(2ma^2)}J_n(\frac{\kappa_{mn}}{a}r)\left\{ \begin{array}{c} \sin n\theta \\ \cos n\theta \end{array} \right. .$$

Now, from the form of the initial condition, we expect that our final solution has no cosine terms and only the $n = 1$ sine term. Hence, we construct a solution of the form

$$\Psi(r, \theta, t) = \sum_{m=1}^{\infty} c_m e^{-i\hbar\kappa_{m1}^2 t/(2ma^2)}J_1(\frac{\kappa_{m1}}{a}r)\sin\theta$$

which, at time $t = 0$, must satisfy

$$\frac{2}{a^2\sqrt{\pi}}r\sin\theta = \sum_{m=1}^{\infty} c_m J_1(\frac{\kappa_{m1}}{a}r)\sin\theta$$

or

$$\frac{2}{a^2\sqrt{\pi}}r = \sum_{m=1}^{\infty} c_m J_1(\frac{\kappa_{m1}}{a}r).$$

From the orthogonality of the Bessel functions, namely, from equation (19.10) in Chapter 12,

$$\int_0^a J_1(\kappa_{m1}r/a)J_1(\kappa_{n1}r/a)r\ dr = a^2\int_0^1 J_1(\kappa_{m1}r)J_1(\kappa_{n1}r)r\ dr = \left\{ \begin{array}{ll} 0, & m \neq n \\ \frac{a^2}{2}J_2^2(\kappa_{n1}), & m = n \end{array} \right.$$

we have

$$\frac{2}{a^2\sqrt{\pi}}\int_0^a J_1(\kappa_{n1}r/a)r^2\ dr = c_n\frac{a^2}{2}J_0^2(\kappa_{n1}).$$

Note that, by changing variables and using equation (15.1) in Chapter 12,

$$\int_0^a J_1(\kappa_{n1}r/a)r^2\ dr = \frac{a^3}{\kappa_{n1}^3}\int_0^{\kappa_{n1}} J_1(x)x^2\ dx = \frac{a^3}{\kappa_{n1}^3}\int_0^{\kappa_{n1}} \frac{d}{dx}\left(x^2 J_2(x)\right)\ dx = \frac{a^3}{\kappa_{n1}}J_2(\kappa_{n1})$$

so

$$c_n = \frac{4}{a\kappa_{n1}\sqrt{\pi}J_2(\kappa_{n1})}$$

and

$$\Psi(r,\theta,t) = \frac{4}{a\sqrt{\pi}}\sum_{m=1}^{\infty}\frac{1}{\kappa_{m1}J_2(\kappa_{m1})}e^{-i\hbar\kappa_{m1}^2 t/(2ma^2)}J_1(\frac{\kappa_{m1}}{a}r)\sin\theta.$$

3. Problem #5 on pages 662-663 of the text.

The displacement $y(x,t)$ of the wire satisfies the initial-boundary value problem

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\\ y(0,t) &= 2\sin 3t\\ y(x,0) &= 0\\ \frac{\partial y}{\partial t}(x,0) &= 0.\end{aligned}$$

Defining $Y(x,p)$ as the Laplace transform of $y(x,t)$ with respect to t:

$$Y(x,p) = \int_0^{\infty} e^{-pt}y(x,t)dt$$

leads to the differential equation

$$\begin{aligned}\mathcal{L}(\frac{\partial^2 y}{\partial x^2}) &= \frac{1}{v^2}\mathcal{L}(\frac{\partial^2 y}{\partial t^2})\\ \frac{\partial^2 Y}{\partial x^2} &= \frac{1}{v^2}\left(p^2 Y - py(x,0) - \frac{\partial y}{\partial t}(x,0)\right)\\ \frac{\partial^2 Y}{\partial x^2} &= \frac{p^2}{v^2}Y \qquad (1)\end{aligned}$$

with initial condition

$$Y(0,p) = L(2\sin 3t) = \frac{6}{9+p^2}. \qquad (2)$$

The solution of equation (1) is

$$Y(x,p) = c_1 e^{-px/v} + c_2 e^{px/v}.$$

Since the second term in $Y(x,p)$ has no inverse Laplace transform we take $c_2 = 0$. Imposing the initial condition (2) then gives

$$Y(x,p) = \frac{6}{9+p^2}e^{-px/v}.$$

From the shift theorem for Laplace transforms (see L28),

$$y(x,t) = 2\sin 3(t-\frac{x}{v})H(t-\frac{x}{v})$$

where H is the Heaviside step function:

$$H(x) = \begin{cases} 1, & x > 0\\ 0, & x < 0\end{cases}$$

that is,

$$y(x,t) = \begin{cases} 2\sin 3(t-\frac{x}{v}), & vt > x\\ 0, & vt < x\end{cases}.$$

4. Problem #3 on page 658 of the text.

If the charge q is located at the point $\overrightarrow{r}_a = (0,0,a)$ inside the grounded sphere ($a < R$), the solution $V(r,\theta,\phi)$ of

$$\Delta V = -4\pi q \delta(\overrightarrow{r} - \overrightarrow{r}_a)$$

with the boundary condition

$$V(R,\theta,\phi) = 0$$

can be found in the form of a sum $V = v + w$ where

$$\Delta v = -4\pi q \delta(\overrightarrow{r} - \overrightarrow{r}_a)$$

inside the sphere, and

$$\Delta w = 0$$

inside the sphere with the boundary condition

$$w(R,\theta,\phi) = -v(R,\theta,\phi).$$

We know (see (8.14) in Chapter 13) that

$$v(r,\theta,\phi) = \frac{q}{|\overrightarrow{r} - \overrightarrow{r}_a|} = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} \tag{3}$$

so we need to find w as a solution of Laplace's equation inside the sphere and with the boundary condition

$$w(R,\theta,\phi) = -\frac{q}{\sqrt{R^2 - 2aR\cos\theta + a^2}}.$$

Using the solutions for Laplace's equation in spherical coordinates (see (8.15) in Chapter 13), and that w must be well-behaved at $r = 0$ and cannot depend on ϕ due to symmetry, we obtain

$$w(r,\theta,\phi) = \sum_{l=0}^{\infty} c_l r^l P_l(\cos\theta).$$

The boundary condition implies

$$\sum_{l=0}^{\infty} c_l R^l P_l(\cos\theta) = -\frac{q}{\sqrt{R^2 - 2aR\cos\theta + a^2}}.$$

Now, we cannot use formula (8.18) in Chapter 13 because the series does not converge if $a < R$. However, we can simply interchange R and a in that formula to obtain

$$\frac{q}{\sqrt{R^2 - 2aR\cos\theta + a^2}} = q\sum_{l=0}^{\infty} \frac{a^l}{R^{l+1}} P_l(\cos\theta) \tag{4}$$

for $a < R$. Hence,

$$c_l = -\frac{qa^l}{R^{2l+1}}$$

and

$$w(r,\theta,\phi) = -q\sum_{l=0}^{\infty} \frac{a^l}{R^{2l+1}} r^l P_l(\cos\theta).$$

This series can actually be summed by rewriting it and using (4):

$$\begin{aligned}
-q\sum_{l=0}^{\infty} \frac{a^l}{R^{2l+1}} r^l P_l(\cos\theta) &= -qR\sum_{l=0}^{\infty} \frac{(ar)^l}{(R^2)^{l+1}} P_l(\cos\theta) \\
&= -\frac{qR}{\sqrt{(R^2)^2 - 2\,(ar)\,(R^2)\cos\theta + (ar)^2}} \\
&= -\frac{Rq/a}{\sqrt{(R^2/a)^2 - 2r(R^2/a)\cos\theta + r^2}} \\
&= -\frac{Rq/a}{\sqrt{r^2 - 2r(R^2/a)\cos\theta + (R^2/a)^2}}
\end{aligned}$$

and, finally, we have

$$V(r,\theta,\phi) = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} - \frac{Rq/a}{\sqrt{r^2 - 2r(R^2/a)\cos\theta + (R^2/a)^2}}.$$

Comparing with (3) we see that this last result is the same as the potential arising from a charge of magnitude $-Rq/a$ located at $(0, 0, R^2/a)$.

5. Show that the curve between the points (r_1, θ_1) and (r_2, θ_2) that minimizes

$$I = \int_{(r_1,\theta_1)}^{(r_2,\theta_2)} \frac{1}{r}\, ds$$

where

$$ds = \sqrt{dr^2 + r^2 d\theta^2}$$

is a logarithmic spiral.

Setting

$$I = \int_{r_1}^{r_2} \frac{1}{r}\sqrt{1 + r^2(\theta')^2}\, dr = \int_{r_1}^{r_2} F(\theta, \theta', r)\, dr$$

we have that I is stationary if

$$\frac{d}{dr}\frac{\partial F}{\partial \theta'} - \frac{\partial F}{\partial \theta} = 0$$

or,

$$\frac{d}{dr}\frac{\theta'}{r\sqrt{1 + r^2(\theta')^2}} = 0$$

so

$$\frac{r\theta'}{\sqrt{1 + r^2(\theta')^2}} = c$$

where c is a constant. Solving for θ',

$$\theta' = \pm\frac{c}{r\sqrt{1 - c^2}}.$$

Setting $K = \pm\frac{c}{\sqrt{1-c^2}}$ we have

$$\frac{d\theta}{dr} = \frac{K}{r} \Rightarrow \theta = K\log r + C \Rightarrow r = e^{(\theta - C)/K}$$

which we recognize as a logarithmic spiral of the form $r = ae^{b\theta}$.