Math/Phys 4530 Midterm Exam

Fall 2010

Dr. Robinson Name ______________________________

Part II. Do all of the problems below. Turn in this test page with your work.

1. Consider the function
$$f(x) = \begin{cases} 1, & -\pi \le x < 0 \\ 0, & 0 < x \le \pi \end{cases} .$$
   (a) Expand $f(x)$ in a Fourier series in sine and cosine functions, that is, as a series of the form
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$
   (b) Expand $f(x)$ as a series of complex exponentials, that is, in a series of the form
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

2. The Laplace transform of the function $f(x)$ is defined, as usual, by
$$\mathcal{L}(f) = F(p) = \int_0^{\infty} e^{-px} f(x)\, dx.$$
   (a) Use the table of Laplace transforms to show that taking the Laplace transform of the Laguerre differential equation
$$xy''(x) + (1-x)y'(x) + \lambda y = 0 \tag{1}$$
   results in the first order differential equation
$$Y'(p) = \frac{1+\lambda-p}{p(p-1)} Y(p) \tag{2}$$
   where $Y(p)$ is the Laplace transform of $y(x)$. (Hint: by L32 on the table, $\mathcal{L}(xf) = -\frac{d}{dp}\mathcal{L}(f)$).
   (b) The general solution of (2) is
$$Y(p) = c\frac{(p-1)^{\lambda}}{p^{1+\lambda}}.$$
   Use this result and the inverse transform to find a solution of (1) if $\lambda = 2$.

3. On the take-home portion of the exam you showed that the Chebyshev polynomials, the first few of which are

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \end{aligned}$$

satisfied the orthogonality relation

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = 0 \text{ if } m \neq n.$$

(a) Show that the change of variables $x = \cos\theta$ results in the orthogonality condition

$$\int_0^\pi T_m(\cos\theta)T_n(\cos\theta)\, d\theta = 0 \text{ if } m \neq n. \tag{3}$$

(b) It is easily seen that

$$\int_0^\pi \left(T_0(\cos\theta)\right)^2 d\theta = \pi$$

and it can be shown that

$$\int_0^\pi \left(T_n(\cos\theta)\right)^2 d\theta = \frac{\pi}{2} \text{ if } n \neq 0.$$

Using these two relations and (3) find the values of c_0 and c_1 in the expansion

$$f(\theta) = \sum_{n=0}^{\infty} c_n T_n(\cos\theta)$$

for

$$f(\theta) = \begin{cases} 1, & 0 \leq \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta \leq \pi \end{cases}.$$