Math/Phys 4530 Midterm Exam

Fall 2011

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Part II. Solutions

1. Recall that the functions e^{inx} satisfy the orthogonality condition

$$\int_{-\pi}^{\pi} e^{-imx} e^{inx} dx = \begin{cases} 0, & m \neq n \\ 2\pi, & m = n \end{cases}.$$

Consider the function $f(x)$ defined on $(-\pi, \pi)$ by

$$f(x) = \begin{cases} 1, & |x| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}.$$

(a) Expand $f(x)$ as a series of complex exponentials, that is, in a series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$\int_{-\pi}^{\pi} e^{-imx} f(x)\ dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{-imx} e^{inx} dx = 2\pi c_m$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1\ dx = \frac{1}{2}$$

If $n \neq 0$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\ dx = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-inx}\ dx = \frac{i}{2\pi n}(e^{-in\pi/2} - e^{in\pi/2}) = \frac{\sin\frac{n\pi}{2}}{n\pi}$$

$$f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\frac{n\pi}{2}}{n\pi} e^{inx}.$$

(b) Without doing any further integration, express your series from part (a) as a Fourier cosine series, that is, in a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$\begin{aligned}
f(x) &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\frac{n\pi}{2}}{n\pi} e^{inx} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n\pi} e^{inx} + \sum_{n=-\infty}^{-1} \frac{\sin\frac{n\pi}{2}}{n\pi} e^{inx} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n\pi} e^{inx} + \sum_{n=1}^{\infty} \frac{\sin\frac{-n\pi}{2}}{(-n)\pi} e^{-inx} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n\pi} (e^{inx} + e^{-inx}) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\sin\frac{n\pi}{2}}{n\pi} \cos nx
\end{aligned}$$

2. Recall that, if f is an odd function on $(-L, L)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\ dx = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi}{L}x)\ dx.$$

Also recall that at the end of the last lecture we were faced with the problem of finding constants c_n such that

$$100 = \sum_{n=1}^{\infty} c_n \sinh(n\pi)\sin(\frac{n\pi}{L}x)$$

for x in the interval $(0, L)$. Find the constants c_n.

Let $c_n \sinh(n\pi) = b_n$

$$100 = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$$

$$b_n = \frac{2}{L}\int_{0}^{L} 100\sin(\frac{n\pi}{L}x)\ dx = \frac{200}{L}(-\frac{L}{n\pi})(\cos(n\pi) - 1) = \frac{200}{n\pi}(1 - (-1)^n)$$

$$c_n = \frac{b_n}{\sinh(n\pi)} = \frac{200}{n\pi\sinh(n\pi)}(1 - (-1)^n)$$

3. Consider the initial value problem

$$\begin{aligned} \frac{d^2y}{dt^2} + \omega^2 y &= f(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

where ω is a positive constant.

(a) Let $Y(p)$ and $F(p)$ denote the Laplace transforms of $y(t)$ and $f(t)$, respectively. Show that

$$Y(p) = \frac{F(p)}{p^2 + \omega^2}.$$

$$\begin{aligned} \mathcal{L}(\frac{d^2y}{dt^2} + \omega^2 y) &= \mathcal{L}(f(t)) \\ \mathcal{L}(\frac{d^2y}{dt^2}) + \omega^2\mathcal{L}(y) &= \mathcal{L}(f(t)) \\ p^2Y(p) + \omega^2 Y(p) &= F(p) \\ Y(p) &= \frac{F(p)}{p^2 + \omega^2} \end{aligned}$$

(b) Using the result above, write the solution $y(t)$ as a convolution integral involving a trigonometric function and the yet unspecified function $f(t)$.

$$Y(p) = F(p)\frac{1}{p^2+\omega^2} = F(p)G(p)$$

$$G(p) = \mathcal{L}(g(t)) = \mathcal{L}(\frac{1}{\omega}\sin \omega t)$$

$$y(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau = \frac{1}{\omega}\int_0^t f(\tau)\sin\omega(t-\tau)\ d\tau$$

(c) Evaluate the convolution integral in part (b) for the case where

$$f(t) = \begin{cases} 1, & t < t_0 \\ 0, & t > t_0 \end{cases}$$

where t_0 is a positive constant.

If $t < t_0$

$$\int_0^t f(\tau)\sin\omega(t-\tau)\ d\tau = \int_0^t \sin\omega(t-\tau)\ d\tau = \frac{1-\cos\omega t}{\omega}$$

If $t > t_0$

$$\int_0^t f(\tau)\sin\omega(t-\tau)\ d\tau = \int_0^{t_0} \sin\omega(t-\tau)\ d\tau = \frac{\cos\omega(t-t_0)-\cos\omega t}{\omega}$$

(d) Evaluate the convolution integral from part (b) in the case where $f(t) = \delta(t-t_0)$ where t_0 is a positive constant.

If $t < t_0$

$$\int_0^t f(\tau)\sin\omega(t-\tau)\ d\tau = \int_0^t \delta(t-t_0)\sin\omega(t-\tau)\ d\tau = 0$$

If $t > t_0$

$$\int_0^t f(\tau)\sin\omega(t-\tau)\ d\tau = \int_0^{t_0} \delta(t-t_0)\sin\omega(t-\tau)\ d\tau = \sin\omega(t_0-\tau)$$

4. Substitute the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

into the differential equation

$$(1-x^2)y'' - 3xy' + \lambda(\lambda+2)y = 0$$

and show that the equation has a polynomial solution if λ is a nonnegative integer.

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} a_n n x^{n-1} \\ y''(x) &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} \end{aligned}$$

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\begin{aligned}
(1-x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 3x\sum_{n=0}^{\infty} a_n n x^{n-1} + \lambda(\lambda+2)\sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} 3a_n n x^n + \lambda(\lambda+2)\sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} 3a_n n x^n + \lambda(\lambda+2)\sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) - a_n(n(n+2) - \lambda(\lambda+2))\right) x^n &= 0
\end{aligned}
$$

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a_{n+2}(n+2)(n+1) - a_n(n(n+2) - \lambda(\lambda+2)) = 0
$$

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a_{n+2} = a_n \frac{(n(n+2) - \lambda(\lambda+2))}{(n+2)(n+1)}
$$

If λ is an odd integer then the solution with $a_0 = 0$ is a polynomial while if λ is an even integer the solution with $a_1 = 0$ is a polynomial.