Math/Phys 4530 Midterm Exam

Fall 2011

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Part I. Solutions

1. Problem #33 on page 43 of the text.

   From the figure,

   $$\cos\theta = \frac{R}{R+h}$$

   or

   $$\sec\theta = \frac{R+h}{R} = 1 + \frac{h}{R}.$$

   Now, $h << R$, so θ can be presumed small. We can expand $\sec\theta$ in a Taylor series at $\theta = 0$ and keep only the first two non-zero terms to obtain

   $$\begin{aligned}\sec\theta &\approx \sec 0 + \left(\frac{d}{d\theta}\sec\theta\right)\Big|_{\theta=0}\theta + \frac{1}{2}\left(\frac{d^2}{d\theta^2}\sec\theta\right)\Big|_{\theta=0}\theta^2 \\ &= \sec 0 + (\sec 0 \tan 0)\,\theta + \frac{1}{2}\left(\sec 0 \tan^2 0 + \sec^3 0\right)\theta^2 \\ &= 1 + \frac{1}{2}\theta^2\end{aligned}$$

   so

   $$\frac{h}{R} = \sec\theta - 1 \approx \frac{1}{2}\theta^2.$$

   Substituting $\theta = \frac{s}{R}$ gives

   $$\frac{h}{R} \approx \frac{s^2}{2R^2}$$

   or

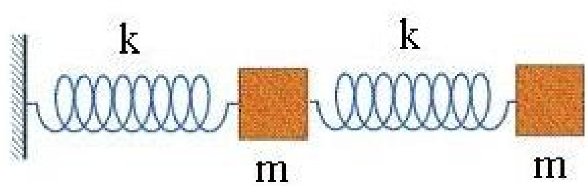
   $$s \approx \sqrt{2Rh}.$$

   Note that, in miles, this is

   $$s \approx \sqrt{2 \times 3960\,\mathrm{mi} \times h\,\mathrm{ft}/(5280\,\mathrm{ft}/\,\mathrm{mi})} = \sqrt{\frac{3h}{2}}\,\mathrm{mi}$$

   where h is dimensionless and with numerical value equal to the height of the tower in feet.

2. Find the characteristic frequencies and describe the characteristic modes of vibration for the system of masses and springs pictured below.



   Let x and y be the coordinates of the two masses at time t relative to their equilibrium positions. The potential energy of the system is

   $$V = \frac{1}{2}kx^2 + \frac{1}{2}k(x-y)^2 = \frac{1}{2}k\left(2x^2 - 2xy + y^2\right)$$

and the equations of motion are

$$\begin{aligned} m\ddot{x} &= -\frac{\partial V}{\partial x} = -\frac{1}{2}k\,(6x-4y) = k(-2x+y) \\ m\ddot{y} &= -\frac{\partial V}{\partial y} = -\frac{1}{2}k\,(-4x+6y) = k(x-y). \end{aligned}$$

Assuming solutions of the form $x = x_0 e^{i\omega t}$ and $y = y_0 e^{i\omega t}$ leads to

$$\begin{aligned} -m\omega^2 x_0 &= k(-2x_0+y_0) \\ -m\omega^2 y_0 &= k(x_0-y_0) \end{aligned}$$

or

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \text{ with } \lambda = \frac{m\omega^2}{k}.$$

The eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

are given by the solutions of

$$\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 1 = 0,$$

namely, $\lambda = (3 \pm \sqrt{5})/2$. The characteristic frequencies are therefore

$$\omega = \sqrt{\frac{\lambda k}{m}} = \sqrt{\frac{(3+\sqrt{5})k}{2m}} \text{ or } \sqrt{\frac{(3-\sqrt{5})k}{2m}}.$$

To find the eigenvectors for we solve the systems

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} (2-\lambda)x_0 - y_0 &= 0 \\ -x_0 + (1-\lambda)y_0 &= 0. \end{aligned}$$

These are equivalent to a single equation (as you can see by multiplying the first of these by $(1-\lambda)$ and the second by $(2-\lambda)$, neither of which are zero) namely,

$$y_0 = (2-\lambda)x_0.$$

For $\lambda = (3+\sqrt{5})/2$ this gives an eigenvector $\left[1, (1-\sqrt{5})/2\right]$ which represents a motion where the masses oscillate in opposite directions but with different amplitudes (the amplitude of the y oscillation is about .6 of that for x). For $\lambda = (3-\sqrt{5})/2$ we have an eigenvector $\left[1, (1+\sqrt{5})/2\right]$ which represents a motion where the masses oscillate back and forth in the same direction with the amplitude of the y oscillation about 1.6 times that of for x.

3. Problem #29 on page 618 of the text. You may take the result of Problem 20 on page 617, namely that the Bessel function of the first kind $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\; d\theta$$

for all integers n, as given.

We first note that

$$\begin{aligned} J_0(x) &= \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\ d\theta \\ &= \frac{1}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\ d\theta + \frac{1}{\pi}\int_{\pi/2}^\pi \cos(x\sin\theta)\ d\theta. \end{aligned}$$

Making the change of variable $u = \pi - \theta$ in the second integral gives

$$\begin{aligned} \int_{\pi/2}^\pi \cos(x\sin\theta)\ d\theta &= \int_{\pi/2}^0 \cos(x\sin(\pi-u))\ (-du) \\ &= \int_0^{\pi/2} \cos(x\sin u)\ du. \end{aligned}$$

Letting $u = \theta$ in this result then leads to

$$J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\ d\theta.$$

Making the change of variable $\alpha = \sin\theta$, $d\alpha = \cos\theta\ d\theta = \sqrt{1-\alpha^2}\ d\theta$ gives

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos(x\alpha)}{\sqrt{1-\alpha^2}}\ d\alpha$$

which we can recognize as an inverse Fourier cosine transform, that is,

$$J_0(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(\alpha)\cos(\alpha x)\ d\alpha$$

where

$$g(\alpha) = \sqrt{\frac{2}{\pi}}\begin{cases} \frac{1}{\sqrt{1-\alpha^2}}, & 0 \le \alpha < 1 \\ 0, & \alpha > 1 \end{cases}.$$

But $g(\alpha)$ is then the Fourier cosine transform of $J_0(x)$, that is,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty J_0(x)\cos(\alpha x)\ dx = g(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1-\alpha^2}}, & 0 \le \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$$

from which it immediately follows by setting $\alpha = 0$ that

$$\int_0^\infty J_0(x)\ dx = 1.$$